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UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

GENERAL THEORY OF INTEGRATION

DEMATH4ELEC7

BLOCK-1

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

GENERAL THEORY OF INTEGRATION

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BLOCK-1 GENERAL THEORY OF INTEGRATION

Introduction to Block-1

The purpose of this book General theory of Integration is to present an exposition of a relatively new theory of the integral (variously called the “generalized Riemann integral”, the “gauge integral”, the “Henstock-Kurzweil integral”, etc.) that corrects the defects in the classical Riemann theory and both simplifies and extends the Lebesgue theory of integration. Not wishing to tell only the easy part of the story, we give here a complete exposition of a theory of integration, initiated around 1960 by Jaroslav Kurzweil and Ralph Henstock.

Although much of this theory is at the level of an undergraduate

course in real analysis, we are aware that some of the more subtle aspects go slightly beyond that level. Hence this monograph is probably most suitable as a text in a first-year graduate course, although much of it can be readily mastered by less advanced students, or a teacher may simply skip over certain proofs. The principal defects in the Riemann integral are several. The most serious one is that the class of Riemann integrable functions is too small.

In block-1 we will learn and understand about Gauges and Integrals, The Riemann integrals, Basic properties of the Integrals, Fundamental theorems of calculus, The Sakes-Henstock lemma.

UNIT -1 GAUGES AND INTEGRALS

STRUCTURE

1.0 Objective
1.1 Introduction
1.2 Length and Tags
1.3 Riemann Sums
1.4 The Right –left procedure
1.5 Various limiting process
1.6 Let us sum up
1.7 Keywords
1.8 Questions for review
1.9 Suggestive readings and references
1.10 Answers to check your progress

1.0 OBJECTIVE

In this unit we will learn and understand about Length and tags, Riemann sums, The right –left procedure, various limiting process.

1.1 INTRODUCTION

The technique of functional integration is used in statistical mechanics, quantum mechanics, control theory, signal processing and many other areas.

This book presents a mathematically rigorous theory of integration and variation in function spaces, including theorems on taking limits under the integral sign. The integration is based on the generalised Riemann integral rather than measure theory, and provides a unified mathematical framework for the diverse applications in theoretical physics and information science.

In particular, this book presents a solution to the long-standing problem of providing a rigorous definition and calculus of the Feynman integral in quantum mechanics.

The symbol R always denotes the real number system, the properties of which we assume to be familiar to the reader. Our principal elementary reference will be the third edition of the book of the author and D.r Sherbert, which will be referred to as [B-S], but there are many other books at that level to which the reader can refer.

Our terminology and notation are standard. If A and B are subsets of a set X, we denote their union by $A \cup B$, their intersection by $A \cap B$, and the relative complement by A-B. If X is understood we sometimes denote X – A by A^c.

We denote the distance between two real numbers x, y by dist $(x,y) := |x-y|$.

It is clear that, for all $x, y, z \in R$, then

(0) dist $(x,y) \geq 0$;

(1) dist$(x,y) = 0$ if and only if $x = y$;

(2) dist $(x,y) =$ dist (y,x);

(3) dist $(x,z) \leq$ dist $(x,y) +$ dist (y,z).

If $x \in R$ and $A \subseteq R$, where $A = \emptyset$, we sometimes write dist $(x,A) :=$ INF {dist$(x,y) : y \in A$} for the distance between x and A. The closed neighborhood of x with radius $r > 0$ is the set $B[x;r] := \{y \in R : |x-y| \leq r\}$,

Which is also called the closed ball with center x and radius r. The open neighborhood of x with radius $r > 0$ is the set

$$B(x;r) := \{y \in R : |x-y| < r\},$$

Which is also called the open ball with center x and radius r.

We are concerned mainly with bounded intervals in R. If $a, b \in R$ and $a \leq b$, we use the notations

$$[a,b] := \{x \in R : a \le x \le b\};$$

$$(a,b) := \{x \in R : a < x < b\};$$

$$[a,b) := \{x \in R : a \le x < b\};$$

$$(a,b] := \{x \in R : a < x \le b\}.$$

The point a is called the left endpoint and the point b is called the right endpoint of each of these intervals. Intervals of the first kind contain both of their endpoints and are called bounded closed intervals, or compact intervals. Intervals of the second kind contain neither of their endpoints and are called bounded open intervals. Intervals of the third and fourth kinds contain exactly one of their endpoints and are called bounded closed -open and bounded open-closed intervals, respectively.

We say that an interval in R is degenerate if it contains at most one point, and that it is nondegenerate if it contains at least two points, in which case it contains infinitely many points. We say that two intervals in R are disjoint if their intersection is empty; that is, if they have no common points. Similarly, we will say that two intervals in R are nonoverlapping if their intersection is either empty or contains at most one point, which is necessarily an endpoint of both intervals.

If $I := [a,b]$ is a nondegenerate compact interval in $R,$ then a partition (or a division) of I is a finite collection $P := \{I_i : i = 1, \ldots\ldots, n\} = \{I_i\}_{i=1}^{n}$ of nonoverlapping compact subintervals I_i such that $I = I_i \cup \ldots\ldots \cup I_n.$ It is always possible to arrange the intervals in increasing order. i.e., such that $\max I_i = \min I_{i+1}$ for $i = 1, \ldots., n-1.$ If we let $x_0 := a$ and $x_i := \max I_i$ for $i = 1, \ldots., n,$ we can write the intervals as

$$I_1 := [x_0, x_1],\ I_2 := [x_1, x_2], \ldots\ldots\ldots, I_n := [x_{n-1}, x_n].$$

Alternatively, we can define a partition P of $I = [a,b]$ by specifying a finite ordered set of points in I:

$$a = x_0 \le x_1 \le x_2 \le \ldots\ldots \le x_n = b,$$

And defining the subintervals I_i by

$$I_i := [x_{i-1}, x_i] \text{ for } i = 1, \ldots., n.$$

Note that n + 1 points of I are required to define a partition P of I into n intervals, that the initial partition point is always the left endpoint of I, and the final partition point is always the right endpoint of I. Furthermore, if the subintevals in P are written in increasing order, then the left endpoint of I_i is the partition point x_{i-1} and the right endpoint of I_i is the partition point x_i for all I = 1,….,n. Ordinarily, we will require that the subintervals in a partition are nondegenerate and that the partition points are distinct, for this can be obtained by simply discarding degenerate subintervals or identical partition points.

In the following we will think of a partition of I as either a collection of nonoverlapping subintervals, or as a finite ordered set of partition points.

1.2 LENGTH AND TAGS

If $I := [a,b]$, with $a \le b$, we define the length of I to be $l(I) := b - a$.

Note that $l(I) \ge 0$, and that $l(I) = 0$ if and only if the endpoints of I coincide. Similarly, the length of any interval having one of the three forms:

$$(a,b), \qquad [a,b), \qquad (a,b]$$

is also defined to be equal to b – a. In particular, $l(\emptyset) = 0$. If $P = \{I_i : i = 1, \ldots., n\}$ is a partition of an interval $I = [a,b]$ such that for each subinterval I_i there is assigned a point $t_i \in I_i$, then we call t_i a tag of I_i. In this case we say that the partition is tagged and we often write

$$P := \{(I_i, t_i) : i = 1, \ldots., n\} = \{(I_i, t_i)\}_{i=1}^{n},$$

or merely $P := \{(I_i, t_i)\}$. Thus, a tagged partition of I is a set of ordered pairs $\{(I_i, t_i) : i = 1, \ldots., n\}$ consisting of intervals I_i that form a partition of I, and points $t_i \in I_i$ that are tags of the intervals I_i. We write a dot over the symbol for a partition to indicate that it is a tagged partition. It is evident that a given partition of I can be tagged in infinitely many ways by choosing different points t_i as tags.

1.3 RIEMANN SUMS

If a function f is defined on a (nondegenerate) compact interval I with values.

In R, we often write $f : I \to R$. If $P = \{(I_i, t_i)\}$ is any tagged partition of I, then the sum

$$S(f : P) := \sum_{i=1}^{n} f(t_i) l(I_i)$$

is called the Riemann sum of f corresponding to $\dot{P}$. If $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, n$, then this Riemann sum has the form

$$S(f : \dot{P}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

It will be familiar to the reader form calculus courses that if $f(x) \geq 0$ for all $x \in I$, then this Riemann sum is an approximation to the "area under the graph of $y = f(x)$".

The Riemann approach to the integral of the function f on I is to define the integral as a "limit" of the Riemann sums as the partitions are taken to be "finer and finer" (in some appropriate sense).

It will be noted that if some of the subintervals I_i in $\dot{P}$ are degenerate (or if some of the partition points coincide), then the corresponding terms in the Riemann sums $S(f; \dot{P})$ vanish. Thus, if we discard these degenerate subintervals, then the value of the Riemann sum is not changed. Consequently, we will ordinarily assume that the subintervals appearing in our partitions are nondegenerate.

1.4 THE RIGHT-LEFT PROCEDURE

In working with Riemann sums, it is sometimes useful to have some (or all) of the tags be endpoints of the subintervals. This can easily be arranged by using the right-left procedure: If $\dot{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ and if the tag t_k is an interior point of the subinterval $[x_{i-1}, x_i]$, then we let $\dot{P}*$ be obtained from $\dot{P}$ by adding the new partition point $\xi := t_k$, so that

$$a = x_0 \leq .... \leq x_{k-1} < \xi < x_k \leq ..... \leq x_n = b.$$

We now tag both subintervals $[x_{k-1}, \xi]$ and $[\xi, x_k]$ by using the tag $t_k = \xi$; hence ξ is the right endpoint of the first of these subintervals, and the left endpoint of the second of these subintervals. We observe that since

$$f(t_k)(x_k - x_{k-1}) = f(t_k)(\xi - x_{k-1}) + f(t_k)(x_k - \xi),$$

Then the Riemann sums $S(f : \dot{P})$ and $S(f : \dot{P}*)$ give the same value. Of course, it is also possible to reverse this process and consolidate two abutting subintervals that have the same point as tag. When we do this, the tag is no longer an endpoint of the resulting subinterval.

Thus, in dealing with tagged partitions, we may assume that:

i. all of the tags are endpoints of the subintervals, or
ii. no tag, except possibly a or b, is an endpoint of the subintervals, or
iii. no point is the tag of two distinct subintervals.

Sometimes it is convenient to make one of these choices.

Subpartitions

By a subpartition we mean a subset of a partition. Similarly, a tagged subparttion is a subset of a tagged partition. If $\dot{Q} = \{([y_{j-1}, y_j], s_j)\}_{j=1}^{m}$ is a tagged subpartition of $[a,b]$, we will also use the notation $S(f : \dot{Q})$ for $\sum_{j=1}^{m} f(s_j)(y_j - y_j - 1)$.

1.5 VARIOUS LIMITING PROCESSES

The precise type of limiting proce4ss that is used to define the integral varies somewhat depending on the text book. The “traditional Riemann method” is to require that the Riemann sums $S(f:\dot{P})$ approach a limit as the maximum length of the subintervals in the partition approaches zero. This method has the advantage that it can also be applied to functions that have their values in the complex number system, or in the finite-dimensional space $\mathbb{R}^n$ (or even in a Banach space). This method is discussed in detail in the third edition of $[B-S]$.

A popular alternative method – often attributed solely to Gaston Darboux (1842-1917), although Giulio Ascoil (1843 – 1896), Henery J.S. Simth (1826-1883) and Karl J. Thomae (1840 – 1921) employed a similar approach in the same year (1875) – is to introduce “lower” and “upper integrals”. The “Darboux method” has certain technical advantages, but it also has at least two disadvantages. One is that it makes heavy use of the order properties of the real number system R, and so extensions to more general values of the function require further treatment. Another disadvantage is that in order to prove that exactly the same class of functions is integrable using the “Darboux approach” as the traditional Riemann approach, it is necessary to prove a rather subtle theorem.

In this book we will not use either the traditional Riemann or the Darboux approach in defining a limit of Riemann sums. Instead we shall employ a limiting process that was recently introduced independently by the Czech mathematician Jaroslav Kurzweil (b.1926) and the English mathematician Ralph Henstock (b. 1923).

This method is slightly more complicated than the Riemann process, yet it yields an integral that is considerably more general and easier to use than the ordinary Riemann integra. It is more general in that the class general in that the class of integrable functions is considerably enlarged, and it is easier to use because it enables one two remove (or at least weaken) certain hypotheses that the Riemann theory requires.

Since we get a lot more with little additional effort, we regard this approach to be a very significant advance.

In the Riemann approach to the integral, the measure of fineness of a partition is given by the maximum length of the subintervals I_i; this means that the lengths of the subintervals are all less thaen or equal to a certain number.

In the Kurzweil-Henstock approach that we adopt, more variation in the lengths of the subintervals is allowes as long as the subintervals over which the functions is "rapidly changing" have "small length".

Thus, in Figure 1.1, we make the approximation of the Riemann sums to the area a close one by taking the length of the intervals I_3 and I_4 small, since fi is increasing rapidly near the right end of $[a,b]$. These is no particular need to make the lengths of I_1 and I_2 small, since the function is nearly constant over the first part of the interval $[a,b]$.

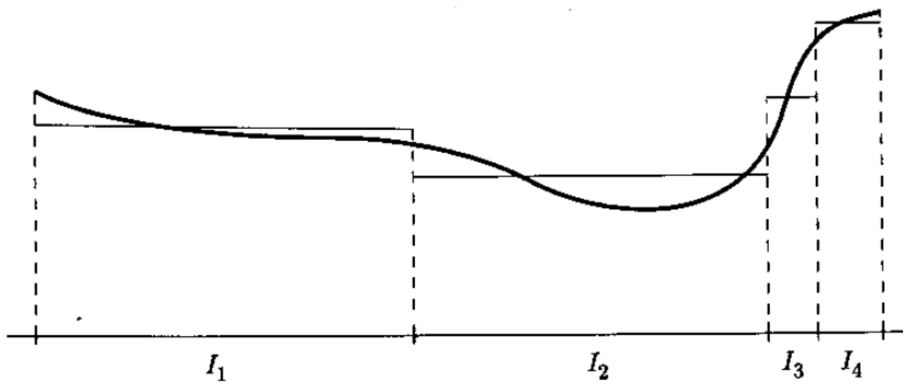

Figure 1.1

Gauges

The Kurzweil -Henstock approach places more attention on the tags than the traditional approach does. In fact, we shall govern the fineness of the tagged partition $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ by requiring that each subinterval I_i is contained in an interval $B[t_i;\delta_i] := [t_i - \delta_i, t_i + \delta_i]$ that depends on the tag t_i. the following definitions will be used.

Definition. If $I := [a,b] \subset R$, then a functions $\delta : I \to R$ is said to be a gauge on I if $\delta(t) > 0$ for all $t \in I$. The interval around $t \in I$ controlled by the gauge δ is the interval $B[t;\delta(t)] := [t - \delta(t), t + \delta(t)]$.

Definition. Let $I := [a,b]$ and let $\dot{P} := \{(I_t, t_i)\}_{i=1}^{n}$ be a tagged partition. If δ is a gauge on I**,** then we say that $\dot{P}$ is $\delta-$fine if

$$I_i \subseteq \left[t_i - \delta(t_i), t_i + \delta(t_i)\right] \text{ for all } i = 1, \ldots., n;$$

that is, if each subinterval I_i is contained in the interval $B\left[t_i; \delta(t_i)\right]$ controlled by the point t_i. (See Figure 1.2.) Sometimes, when the tagged partition $\dot{P}$ is δ-fine, we say that $\dot{P}$ is **subordinate** to δ or write $\dot{P} << \delta$.

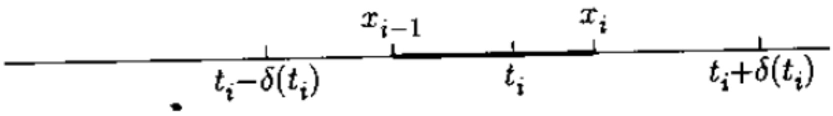

Figure 1.2

Remarks. **(a)** Only a tagged partition can be δ-fine; hence it is not necessary to employ the word "tagged" in referring to δ-fine partitions.

(b) If $I_i := [x_{i-1}, x_i]$, then the partition $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is δ-fine if and only if $t_i - \delta(t_i) \le x_{i-1} \le t_i \le x_i \le t_i + \delta(t_i)$ for all $i = 1, \ldots., n$.

(c) The partition $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is δ-fine if and only if

$$I_i \subseteq B\left[t_i; \delta(t_i)\right] \qquad \text{for all } i = 1, \ldots., n.$$

We now give some examples of gauges that will be instructive and useful.

Examples. (a) If $\delta > 0$ is a positive number, then we can define a gauge $\delta : I \to R$ by setting $\delta(t) := \delta$ for all $x \in I$.

Such a gauge is called a **constant gauge.** We note that a partition

$\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is δ-fine for this constant gauge if and only if

$$I_i \subseteq [t_i - \delta, t_i + \delta] = B[t_i; \delta] \qquad \text{for all } i = 1, \ldots., n.$$

This is readily seen to imply that $l(I_i) \le 2\delta$ for all i.

(b) If δ_1 and δ_2 are two gauges on $I := [a,b]$, and if we define

$\delta(t) := \min\{\delta_1(t), \delta_2(t)\}$ for $t \in I$,

Then it is immediate that δ is a gauge on *I*. Clearly. every partition of ***I*** that is δ-fine is both δ_1-fine and δ_2-fine. This construction can be extended to any finite number of gauges on ***I*.**

(c) It is often convenient to choose a gauge δ that will force a given point to be a tag for any δ-fine partition,

For example, let $I := [0,1]$ and let $\delta(0) := \frac{1}{4}$ and $\delta(t) := \frac{1}{2}t$ for $0 < t \le 1$. Evidently δ is a gauge on *I*. If $\dot{P}$ is a δ-fine partition of *I*, then $0 \in I$ must belong to some subinterval $I_1 = [0, x_1]$ in $\dot{P}$. We claim that the tag t_1 for I_1 must be 0. Indeed, since $\dot{P}$ is δ-fine, we must have $[0, x_1] \subseteq [t_1 - \delta(t_1), t_1 + \delta(t_1)]$ which implies that

$$(1.\alpha) \qquad t_1 - \delta(t_1) \le 0.$$

Now, if $t_1 > 0$, then $\delta(t_1) = \frac{1}{2}t_1$ so that $t_1 - \delta(t_1) = t_1 - \frac{1}{2}t_1 > 0$, contradicting the inequality $(1,\alpha)$. Therefore, we must have $t_1 = 0$, *as* asserted.

We will study this gauge further in the exercises at the end of this section.

(d) Let $a < c < b$ and let δ be a gauge on $[a,b]$. If $\dot{P}'$ is a partition of $[a,c]$ that is δ-fine and if $\dot{P}''$ is a partition of $[c,b]$ that is δ-fine, then $\dot{P}' \cup \dot{P}''$ is a partition of $[a,b]$ that is δ-fine.

(e) Let $a < c < b$ and let δ' and δ'' be gauges on *he* intervals $[a,c]$ and $[c,b]$, respectively. If δ is defined on $[a,b]$ by

$$\delta(t) := \begin{cases} \delta'(t) & \text{if} \quad t \in [a,c), \\ \min\{\delta'(c), \delta''(c)\} & \text{if} \quad t = c, \\ \delta''(t) & \text{if} \quad t \in (c,b], \end{cases}$$

then δ is a gauge on $[a,b]$. Moreover, if $\dot{P}'$ is a partition of $[a,c]$ that is δ'- fine, and $\dot{P}''$ is a partition of $[c,b]$, that is δ''-fine, then $\dot{P}' \cup \dot{P}''$ is a partition of $[a,b]$ that has c as a partition point. However, $\dot{P}' \cup \dot{P}''$ may *not* be δ-fine.

(Why?)

(f) Let δ' and δ'' be as in (d) and let δ^* be defined on [a, b] by

$$\delta^*(t) := \begin{cases} \min\left(\delta'(t), \frac{1}{2}(c-t)\right) & \text{if} \quad t \in [a,c), \\ \min(\delta'(c), \delta''(c)) & \text{if} \quad t = c, \\ \min\left(\delta''(t), \frac{1}{2}(t-c)\right) & \text{if} \quad t \in (c,b]. \end{cases}$$

It is clear that δ^* is a gauge on $[a,b]$, and it is easy to show that every δ^*- fine partition $\dot{P}$ of $[a,b]$ must have *c* as a tag for any subinterval of $\dot{P}$ that contains *c*. Thus, if we use the right-left procedure mentioned above, every δ^*-fine partition $\dot{P}$ of $[a,b]$ gives rise to a partition of $[a,c]$ that is δ'-fine, and to a partition of $[c,b]$ that is δ''-fine.

Some Intuitive Remarks

If I is a compact interval and δ is a gauge on I, we can think that every point $t \in I$ "controls" (or has some "influence on") every point in the closed inter- val $B[t;\delta(t)] = [t-\delta(t), t+\delta(t)]$, and hence on every subinterval contained in this interval. We note that some points in I control large intervals, and other points control very small intervals. The question arises whether, for an arbitrary gauge δ, one can always find a tagged partition

$\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ where each tag t_i controls the corresponding subinterval I_i.

The Existence of δ-Fine Partitions

It will now be shown that if $I := [a,b] \subset R$ is a nondegenerate compact interval, and if δ is any gauge defined on *I*, then there always exist tagged partitions of *I* that are δ-fine.

This result was established and used in the space $R^m, m \geq 1$, by Pierre Cousin (1867-1933). It is a reflection of the compactness of *I* and is sometimes called the "Fineness Theorem".

Cousin's Theorem. If $I := [a,b]$ is a nondegenerate compact interval in R and δ is a gauge on I, then there exists a partition of I that is δ-fine.

Proof. The proof is by contradiction. We suppose that I does not have a δ-fine partition. Now let $c := \frac{1}{2}(a+b)$ and bisect I into: $[a,c]$, $[c,b]$.

We claim that at least one of these subintervals does not have a δ-fine partition; for, if they both have δ-fine partitions, then the union of these partitions would be a δ-fine partition of [a, b], as was noted in Example 1.3(d). We let $I^1 := [a,c]$ if this subinterval does not have any δ-fine partition; otherwise, let $I^1 := [c,b]$. Relabel I^1 as $[a_1, b_1]$, let $c_1 := \frac{1}{2}(a_1 + b_1)$ and bisect I^1 into :

$$[a_1, c_1], \qquad [c_1, b_1].$$

As before, at least one of these subintervals does not have a δ-fine partition. We let $I^2 := [a_1, c_1]$ if it does not have a δ-fine partition; otherwise, let $I^2 := [c_2, b_1]$. Relabel I^2 as $[a_2, b_2]$ and bisect again. In this manner, we

Notes

obtain a sequence (I^n) of compact subintervals of $I=[a,b]$ that is nested in the sense that

$$[a,b]=I\supset I^1\supset .....\supset I^n\supset I^{n+1}\supset .....$$

The Nested Intervals Property (see [B-S; p. 46]) implies that there is a unique number ξ that lies in all of the intervals I^n. However, since $\delta(\xi)>0$, the Archimedean Property of R implies that there exists $p\in N$ such that

$$l(I^p)=(b-a)/2^p<\delta(\xi),$$

Whence $I^p\subset[\xi-\delta(\xi),\xi+\delta(\xi)]$. Therefore the pair (I^p,ξ) is a (trivial) δ-fine partition of I^p. But this is contrary to the construction of the I^n as subintervals of *1* that have no δ-fine partitions.

This contradiction shows that, for every gauge δ on *I*, there exists a δ-fine partition of *I*.

Check Your Progress:

1. Discuss about Riemann sums.

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2. State and prove Cousin's theorem.

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3. Explain about Various limiting process

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1.6 LET US SUM UP

1. Given partition of I can be tagged in infinitely many ways by choosing different points t_i as tags.

2. If a function f is defined on a (nondegenerate) compact interval I with values.

3. In R, we often write $f : I \to R$. If $P = \{(I_i, t_i)\}$ is any tagged partition of I, then the sum $S(f : P) := \sum_{i=1}^{n} f(t_i) l(I_i)$ is called the Riemann sum of f corresponding to $\dot{P}$.

4. In working with Riemann sums, it is sometimes useful to have some (or all) of the tags be endpoints of the subintervals. This can easily be arranged by using the right-left procedure: If $\dot{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ and if the tag t_k is an interior point of the subinterval $[x_{i-1}, x_i]$, then we let $\dot{P}*$ be obtained from $\dot{P}$ by adding the new partition point $\xi := t_k$, so that

$a = x_0 \le .... \le x_{k-1} < \xi < x_k \le ..... \le x_n = b.$

5. Let $I := [a, b]$ and let $\dot{P} := \{(I_t, t_i)\}_{i=1}^{n}$ be a tagged partition. If δ is a gauge on I**,** then we say that $\dot{P}$ is $\delta -$fine if

$$I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] \text{ for all } i = 1, ..., n;$$

that is, if each subinterval I_i is contained in the interval

$B[t_i; \delta(t_i)]$ controlled by the point t_i.

6. if $I := [a, b] \subset R$ is a nondegenerate compact interval, and if δ is any gauge defined on *I*, then there always exist tagged partitions of *I* that are δ-fine.

7. If $I := [a, b]$ is a nondegenerate compact interval in R and δ is a gauge on I, then there exists a partition of I that is δ-fine.

1.7 KEY WORDS

Tagged portions

Arcmedian property

Riemann Sums

Various limiting process

Non degenerate compact interval

1.8 QUESTIONS FOR REVIEW

1. Explain about Riemann sums
2. Explain about the right left procedure
3. Discuss about various limiting process

1.9 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle
2. The elements of Integration and Lebesgue Meassure
3. A course on integration- Nicolas Lerner
4. General theory of Integration- Dr. E.W. Hobson
5. General theory of Integration- P.Muldowney
6. General theory of functions and Integration- Angus E.Taylor

1.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 1.3
2. See section 1.4
3. See section 1.5

UNIT-2 THE RIEMANN AND GENERALIZED RIEMAN INTEGRALS

STRUCTURE

2.0 Objective

2.1 Introduction

2.2 Riemann Integral

2.3 Generalized Riemann integrable on I

2.4 Equivalence theorem

2.5 Uniqueness theorem

2.6 Consistency theorem

2.7 The Lebesque integral

2.8 Let us sum up

2.9 Key words

2.10 Questions for review

2.11 Suggestive readings and references

2.12 Answers to check your progress

2.0 OBJECTIVE

In this unit we will learn and understand about Riemann integral, Equivalence theorem, Uniqueness theorem, The Lebsque Integral.

2.1 INTRODUCTION

In the branch of mathematics known as real analysis, the **Riemann integral**, created by Bernhard Riemann, was the first rigorous definition of the integral of a function on an interval. It was presented to the faculty at the University of Göttingen in 1854, but not published in a journal until 1868.

For many functions and practical applications, the Riemann integral can be evaluated by the fundamental theorem of calculus or approximated by numerical integration.

The Riemann integral is unsuitable for many theoretical purposes. Some of the technical deficiencies in Riemann integration can be remedied with the Riemann–Stieltjes integral, and most disappear with the Lebesgue integral though the latter does not have a satisfactory treatment of improper integrals. The gauge integral is a generalisation of the Lebesgue integral that is at once closer to the Riemann integral. These more general theories allow for the integration of more "jagged" or "highly oscillating" functions whose Riemann integral does not exist; but the theories give the same value as the Riemann integral when it does exist.

We are now prepared to define the integrals. While we will be primarily interested in the (generalized Riemann) integral, we first define the Riemann integral for the purpose of comparison.

2.2 RIEMANN INTEGRAL

Definition. A function $f : I \to R$ is said to be **R-integrable (or Riemann integrable) on** I if there exists a number $A \in R$ such that for every $\varepsilon > 0$ there exists a number $\gamma_\varepsilon > 0$ such that if $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any tagged partition of I such that $l(I_i) \le \gamma_\varepsilon$ for $i = 1,\ldots.,n$, then

$$\left|S(f;\dot{P}) - A\right| \le \varepsilon.$$

The collection of all functions that are R-integrable on an interval I will often be denoted by $R(I)$.

We will now give two definitions of the generalized Riemann integral. The first one differs from Definition 2.3 only in that the constant γ_ε is replaced by a gauge on I; that is, by a function $\gamma_\varepsilon : I \to (0,\infty)$.

2.3 GENERALIZED RIEMANN INTEGRAL ON I

Definition. A function $f : I \to R$ is said to be generalized Riemann integrable on I if there exists a number $B \in R$ such that for every $\varepsilon > 0$ there exists a gauge γ_ε, on I such that if $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any tagged partition of I such that $l(I_i) \le \gamma_\varepsilon(t_i)$ for $i = 1,\ldots\ldots,n$, then

$$(1,\beta) \qquad \left|S(f;\dot{P}) - B\right| \le \varepsilon.$$

In practice, we will use the following definition of the integral, based on the notion of 6-fineness of a partition with respect to a gauge.

Definition. A function $f : I \to R$ is said to be **generalized Riemann integrable on** I if there exists a number $C \in R$ such that for every $\varepsilon > 0$ there exists a gauge δ_ε, on I such that if $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any tagged partition of I that is δ_ε-fine, then

$$(1.\gamma) \qquad \left|S(f;\dot{P}) - C\right| \le \varepsilon.$$

The collection of all functions that are generalized Riemann integrable on an interval I will be denoted by $R^*(I)$.

It would be highly inconvenient if Definitions 2.4 and 2.5 led to different collections of integrable functions, or different values for the integral. We will now show that they do not do so.

2.4 EQUIVALENCE THEOREM

Definitions 2.3 and 2.4 lead to equivalent integrals.

Proof. Suppose that $f : I \to R$ is integrable in the sense of Definition 1.6, so that there exists a number B such that given $\varepsilon > 0$ there exists a gauge γ_ε as in Definition 1.6. We define $\delta_\varepsilon(t) := \frac{1}{2}\gamma_\varepsilon(t)$ for $t \in I$, so that δ_ε is a gauge on I. If $\dot{P} = \{(I_i, t_i)\}_{i=1}^{n}$ is a δ_ε-fine partition of I, then

$$I_i \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)] = \left[t_i - \frac{1}{2}\gamma_\varepsilon(t_i), t_i + \frac{1}{2}\gamma_\varepsilon(t_i)\right],$$

Whence $l(I_i) \le \gamma_\varepsilon(t_i)$ for all $i = 1,\ldots,n.$ consequently the condition in Definition 2.4 is satisfied and so inequality $(1.\beta)$ holds. We have shown that if $\dot{P}$ is any δ_ε -fine partition of I, then $\left|S(f;\dot{P}) - B\right| \le \varepsilon.$ Since $\varepsilon > 0$ is arbitrary, then fi is integrable in the sense fo Definition 1.7 with C = B.

Conversely, suppose that Definition 1.7 is satisfied, so there exists a number C such that given $\varepsilon > 0$ there exists a gauge δ_ε as in Definition 2.7. We define $\gamma_\varepsilon(t) := \delta_\varepsilon(t),$ so that γ_ε is a gauge on I. If the partition $\dot{P} = \{(I_i, t_i)\}_{i=1}^{n}$ satisfies

$$l(I_i) \le \gamma_\varepsilon(t_i) = \delta_\varepsilon(t_i),$$

Then $I_i \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)]$ for all $i = 1,\ldots.,n,$ so that $\dot{P}$ is $\delta_\varepsilon -$fine. Consequently the condition in Definition 1.7 is satisfied and so inequality $(1.\gamma)$ holds. We have shown that if $\dot{P}$ is any partition of I with $l(I_i) \le \gamma_\varepsilon(t_i)$ for all $i = 1,\ldots.,n,$ then $\left|S(f;\dot{P}) - C\right| \le \varepsilon.$ Since $\varepsilon > 0$ is arbitrary, then f is integrable in the sense of Definition 1.6 with B = C.

Theorem 2.5 shows that Definitions 2.3 and 2.4 give the same collection of generalized Riemann integrable functions and the same value for the integral. It is important to know that the number C in Definition 2.4 is uniquely determined (when it exists). Because of the importance of this uniqueness result, we give its proof here.

2.5 UNIQUENESS THEOREM

There is at most one number C that satisfies the property in Definition 2.4.

Proof. Suppose $C' \neq C''$ and let $\varepsilon := \frac{1}{3}\left|C' \neq C''\right| > 0$. If C' satisfies Definition 2.4. then there exists a gauge δ_{ε}^{I} on I such that if $\dot{P}$ is a δ_{ε}^{I}-fine partition of *I*, then $\left|S\left(f;\dot{P}\right) - C'\right| \leq \varepsilon$. Similarly, if C'' satisfies Definition 2.4, there exists a gauge $\delta_{\varepsilon}^{"}$ on *I* such that if $\dot{P}$ is a $\delta_{\varepsilon}^{"}$-fine partition of *I*, then $\left|S\left(f;\dot{P}\right) - C''\right| \leq \varepsilon$. Now let $\delta_{\varepsilon} :=$min $\left\{\delta_{\varepsilon}', \delta_{\varepsilon}''\right\}$ so that δ_{ε} is a gauge on *I* and let $\dot{P}$ be a δ_{ε}-fine partition of *I*. Then the partition $\dot{P}$ is both δ_{ε}'-fine and δ''-fine. Using the Triangle Inequality, we have

$$\left|C' - C''\right| \leq \left|C' - S\left(f;\dot{P}\right)\right|\left|S\left(f;\dot{P}\right) - C''\right| \leq \varepsilon + \varepsilon < \left|C' - C''\right|,$$

which is a contradiction. Q.E.D. The next result is a formal statement of the fact that the Riemann integral is contained in the generalized Riemann integrals of Definitions 2.4 and 2.5.

2.6 CONSISTENCY THEOREM:

Let $I := [a,b]$ be a compact interval in R and let $f : I \to R$. If f is R-integrable on I, then f is also integrable on I in the sense of Definitions 2.4 and 2.5, and the integrals are equal.

Proof. It is immediate that the Riemann integral is a special case of the integral in Definition 2.4. Since we have seen that the integrals in Definitions 2.4 and 2.5 are equivalent, the assertion follows.Q.E.D.

Notes

Remarks. (a) In the following we will discuss mainly the generalized Riemann integral. To simplify our terminology, unless there is specific mention to the contrary, the words "integral", "integrable", etc., refer to the generalized Riemann integral of Definitions 1.6 and 1.7. When other notions of the integral are intended, they will be specifically mentioned; in this connection we may refer to the generalized Riemann integral as the R*-integral.

(b) The Consistency Theorem asserts that if a function is R-integrable, then the values of the R-integral and the R*-integral are equal. Thus we may safely denote the value of the integrals of such a function by the same notation. Therefore, we will also denote the R*-integral by one of the symbols:

$$\int_I f \qquad \text{or} \qquad \int_a^b f.$$

In case it is useful to denote the variable, we will employ the notation:

$$\int_a^b f(x)\,dx \qquad or \qquad \int_a^b f(u)\,du.$$

This "calculus notation" is useful when we are dealing with a function that depends on several parameters. It is also useful in connection with the Substitution Theorems that will be discussed in Section 13. However, its use in the Substitution Theorem has the danger that one sometimes resorts to a blind "juggling of symbols", rather than a careful application of a theorem.

More on Terminology

The definition of the integral given by the great German mathematician Bernhard Riemann (1826-1866) is essentially Definition 2.3. The Definition 2.5 was independently by Kurzweil and Henstock; therefore it would be entirely appropriate to call this integral the "Kurzweil-Henstock integral" and some authors use this terminology (or some version of it) — others call it the "gauge integral" , etc.

However, it is a remarkable fact that the integral in Definitions 2.4 and 2.5 also coincides with integrals that were introduced in 1912 by Arnaud Denjoy (1884-1974) and in 1914 by Oskar Perron (1880-1975), although the definitions given by these authors were very different. Thus, it would be appropriate to use the term "Denjoy- Perron-Kurzweil-Henstock integral". Since that name is quite unwieldy, we will merely say "the integral", or the "generalized Riemann integral" as we have stated above.

Why do Gauges Work?

Before we get down to a development of the integral, it is appropriate that we attempt to give an answer to the question: Why do nonconstant gauges work better than constant gauges? We have already attempted to suggest a reason by our Figure 1.1 and the accompanying discussion. We will now expand somewhat on that discussion.

(A) A gauge permits one to enclose a finite or countable set of points in a union of intervals that has small total length and so does not contribute much to the Riemann sums.

For example, let $f:[0,1]\to R$ be Dirichlet's function defined by $f(x):=1$ if $x\in[0,1]$ is rational, and $f(x):=0$ if $x\in[0,1]$ is irrational. Although this function is not R-integrable (see[B-S;p.204]), it will be seen in Example 2.2(b) that f is R*-integrable on [0,1] with integral equal to 0. The proof involves showing that a gauge δ_ε can be constructed that will make the Riemann sums for nay δ_ε-fine partition less that ε.This is accomplished by taking δ_ε appropriately small at the rational points in $[0,1]$. It does not matter how we define δ_ε, at the irrational numbers, since $f(t_i)=0$ when t_i is irrational, so these terms make a zero contribution to the Riemann sums.

(B) A gauge can force one to take a particular point as a tag. This can be useful when a particular point is a source of difficulty; by choosing it as a tag, one can sometimes control the difficulty.

For example, let $g(x) := 1/\sqrt{x}$ when $x \in (0,1]$, so that $g(x) \to \infty$ $as\, x \to 0$. To have the function defined on all of $[0,1]$ we define g(0) := 0. Since the function g is not bounded, it cannot be R-integrable. *If* we use a gauge such as that in Example 1.3(c) that forces the first tag $t_1 = 0$, then the first term in any corresponding Riemann sum will be 0. Hence the function g on the remaining part of the interval $[x_1, 1]$ will be bounded and continuous, and is more easily handled.

(C) The use of gauges gives an improved Fundamental Theorem of Calculus for the R*-integral.

Suppose that $F:[a,b] \to R$ has a derivative $f(t)$ at every point $t \in [a,b]$. Then, by the definition of the derivative at $t \in [a,b]$, given $\varepsilon > 0$ there exists $\delta_\varepsilon(t) > 0$ such that if $0 < |x - t| < \delta_\varepsilon(t), x \in [a,b]$, then

$$\left| \frac{F(x) - F(t)}{x - t} - f(t) \right| \le \varepsilon.$$

Hence the existence of the derivative on *I* provides the existence of a gauge δ_ε on *I*. It will be shown in Section 4 that if $\dot{P}$ is a partition of $[a,b]$ that is δ_ε-fine, the $\left|F(b) - F(a) - S(f;\dot{P})\right| \le \varepsilon(b-a)$. Since $\varepsilon > 0$ is arbitrary, this implies that the derivative $f = F'$ is R*-integrable on $[a,b]$ to the value $F(b) - F(a)$. This argument does not require the assumption that *f is* R- (or R*-)integrable.

2.7 THE LEBESGUE INTEGRAL

In fact, there is one more theory of integration that we will discuss in this book; namely the one that was introduced in 1902 by the French mathematician Henri Lebesgue (1875-1941). This integral, which we will call the L-integral or the Lebesgue integral, was introduced to correct certain "defects" in the R-integral and it has been largely, though not totally, successful. Certainly the L-integral is the main integral used in modern mathematical research, so a serious student of mathematics needs to be- come familiar with it.

However, the L-integral also has certain drawbacks that we believe are largely removed by the R*-integral. Moreover, although there are a number of different approaches to the L-integral, most of them require the investment of a considerable amount of time and effort in developing the notion of the "measure" of certain subsets of R. For that reason, the L-integral is usually regarded as being beyond the reach of most undergraduate students of mathematics, and it is very largely avoided by almost all physicists and engineers. [However, if one is content to work with the L-integral of a function defined on an abstract measure space, as is done in the theory of probability, then the basic features of the L-integral are relatively elementary $\left(see\left[B-1\right]\right).\Big]$

It is a fact that every function that is L-integrable on $\left[a,b\right]$ is R^*- integrable. Of course, we cannot give a proof of this assertion without either giving a definition of the L-integral or using some of its properties. However, it may be interesting to know that E. J. McShane [McS-2, McS-3] has given a (surprising) definition of the L-integral that makes it clear that the L-integral is a special case of the R^*-integral. His modification is to use Definition 1.7 with the only change being that he does not require the tags t_i to belong to the subintervals I_i but only to I; however, he continues to re- quire the intervals I_i It to be contained in the intervals controlled by the gauge at

Notes

t_i. Clearly, if Definition 1.7 is satisfied for all δ_ε-fine Riemann auras when the tags are not required to belong to the subintervals, then this definition is also satisfied by these Riemann sums when the tags are required to belong to these subintervals. Since it is easier for a function to be R*-integrable than to be L-integrable, the L-integral is contained in the R*-integral.

In view of the importance of the L-integral, the question arises whether one can identify the L-integrable functions among the R*-integrable ones. We will show later that the answer is affirmative and that the test is very simple: A function f is L-integrable if and only if both f and its absolute value If $|f|$ are R*-integrable.

Check Your Progress

1. Prove Equivalence theorem.

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2. Prove Uniqueness theorem

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3. Prove Consistency theorem

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2.8 LET US SUM UP

1. A function $f: I \to R$ is said to be R-integrable (or Riemann integrable) on I if there exists a number $A \in R$ such that for every $\varepsilon > 0$ there exists a number $\gamma_\varepsilon > 0$ such that if

$\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any tagged partition of I such that $l(I_i) \le \gamma_\varepsilon$ for $i = 1, \ldots, n$, then $\left|S(f;\dot{P}) - A\right| \le \varepsilon$.

2. The collection of all functions that are generalized Riemann integrable on an interval I will be denoted by $R^*(I)$.

3. A function $f : I \to R$ is said to be generalized Riemann integrable on I if there exists a number $C \in R$ such that for every $\varepsilon > 0$ there exists a gauge δ_ε, on I such that if $\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ is any tagged partition of I that is δ_ε-fine, then

$$(1.\gamma) \qquad \left|S(f;\dot{P}) - C\right| \le \varepsilon.$$

2.9 KEY WORDS

Riemann integral

Uniqueness theorem

Equivalence theorem

The Lebesque Integral

2.10 QUESTIONS FOR REVIEW

1. Explain about Generalized Riemann integral.

2. Prove Uniqueness theorem.

3. Explain about the Lebesque integral.

2.11 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

2.12 ANSWERS TO CHECK YOUR PROGRESS

1. See section 2.5

2. See section 2.6

3. See section 2.7

UNIT-3 BASIC PROPERTIES OF THE INTEGRAL

STRUCTURE

3.0 Objective

3.1 Introduction

3.2 Theorems on integrals

3.3 The integral as a Function of Intervals

3.4 The Squeeze Theorem

3.5 Characterization of regulated functions

3.6 Let us sum up

3.7 Keywords

3.8 Questions for review

3.9 Suggestive readings and references

3.10 Answers to check your progress

3.0 OBJECTIVE

In this unit we will learn and understand about theorems on integrals, The integral as a Function of Intervals, The Squeeze Theorem, Characterization of regulated functions

3.1 INTRODUCTION

We will now establish the most important elementary properties of the (generalized Riemann) integral. Since they are formally the same as for the R-integral, the reader will find them quite familiar. E ven the proofs are only slightly

Notes

different from those for the R-integral, so the reader should find most of this section easy reading.

In this section $i := [a,b]$ denotes any compact interval in . However, it will be shown in Part 2 that most (but not all) of the results presented here remain true for infinite intervals having one of the forms $[a,\infty],[-\infty,b]$, or $[-\infty,\infty]$. For the sake of future convenience, we will mark those theorems that remain valid with *no change* in statement by the symbol $\bullet$, and those that require only a *minor change* in their statements by o. However, sometimes a change or supplementary argument is needed in the proof for these infinite intervals.

Although we will be considering functions with values in R, some of the exercises will consider functions with values in the complex field R. However, it is convenient *not* to permit the functions to take on the extended real values $-\infty$ and ∞.

3.2 THEOREMS ON INTEGRALS

*3.1 Theorem. (a) If f and g are integrable on I to R, then their sum $f+g$ is also integrable on I and

$$(3.\alpha) \qquad \int_I (f+g) = \int_I f + \int_I g.$$

(b) If f is integrable on I and $c \in R$, then *cf* is integrable on I and

$$(3.\alpha) \qquad \int_I cf = c\int_I f.$$

Proof. (a) Let A,B denote the integrals of f,g, respectively. Given $\varepsilon > 0$, let $\delta'_\varepsilon, \delta''_\varepsilon$ be gauges on I such that if the partition $\dot{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is δ'_ε-fine, then

$$\left|S(f;\dot{P}) - A\right| \le \frac{1}{2}\varepsilon,$$

and if $\dot{P}$ is δ''-fine, then

$$\left|S\left(g;\dot{P}\right)-B\right|\leq\frac{1}{2}\varepsilon.$$

Now let $\delta_e(t):=\min\{\delta'_\varepsilon(t),\delta''_\varepsilon(t)\}$ so that if a partition $\dot{P}$ is δ_ε-fine, then it is both δ'-fine and δ''-fine. Since it is easily seen that

$$S\left(f+g;\dot{P}\right)=\sum_{i=1}^{n}(f+g)(t_i)(x_i-x_{i-1})$$

$$=\sum_{i=1}^{n}f(t_i)(x_i-x_{i-1})+\sum_{i=1}^{n}g(t_i)(x_i-x_{i-1})$$

$$=S\left(f;\dot{P}\right)+S\left(g;\dot{P}\right),$$

It is clear that

$$\left|S\left(f+g;\dot{P}\right)-[A+B]\right|=\left|\left[S\left(f;\dot{P}\right)-A\right]+\left[S\left(g;\dot{P}\right)-B\right]\right|$$

$$\leq\left|S\left(f;\dot{P}\right)-A\right|+\left|S\left(g;\dot{P}\right)-B\right|$$

$$\leq\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, then $f+g\in R^*(I)$ with integral $A+B.$

(b) We leave the proof of this assertion to the reader.

By using mathematical induction, we can extend Theorem 3.1 to the case of a linear combination of functions in R*(I). (See Exercise 3.B.)

- **3.2 Theorem.** If $f\in R^*(I)$ and $f(x)\geq 0$ for all $x\in I,$ then

$$(3.\gamma)\qquad \int_I f\geq 0.$$

Proof. Let δ_ε be a gauge on I such that for nay partition $\dot{P}<<\delta_\varepsilon,$ we have $\left|S\left(f;\dot{P}\right)-\int_I f\right|\leq\varepsilon.$ Since $f(x)\geq 0$ for all $x\in I,$ then

$$S\left(f;\dot{P}\right)=\sum_{i=1}^{n}f(t_i)(x_i-x_{i-1})\geq 0.$$

Proof. $(\Rightarrow)$ if $f\in R^*(I)$ with integral A, let $\eta_\varepsilon:=\delta_{\varepsilon/2}>0$ be a gauge on I such that if $\dot{P},\dot{Q}<<\eta_\varepsilon.$ then

$$\left|S\left(f;\dot{P}\right)-A\right|\le\frac{1}{2}\varepsilon \qquad \text{and} \qquad \left|S\left(f:\dot{Q}\right)-A\right|\le\frac{1}{2}\varepsilon.$$

Consequently, we conclude that for such partitions $\dot{P}$ and $\dot{Q}$, then

$$\left|S\left(f;\dot{P}\right)-S\left(f:\dot{Q}\right)\right|\le\left|S\left(f;\dot{P}\right)-A\right|+\left|S\left(f:\dot{Q}\right)-A\right|\le\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon.$$

$(\Leftarrow)$ For each $n\in R$, let δ_n be a gauge on I such that if $\dot{P},\dot{Q}\ll\delta_n$, then

$$\left|S\left(f;\dot{P}\right)-S\left(f:\dot{Q}\right)\right|\le 1/n.$$

Evidently we may assume that these gauges satisfy $\delta_n(t)\ge\delta_{n+1}(t)$ for $t\in I, n\in N$; otherwise, we replaces δ_n by the gauge $\delta_n^{'}(t):=\min\{\delta_1(t),....,\delta_n(t)\}.$

For each $n\in R$, let $\dot{P}_n\ll\delta_n.$ Clearly, if $m>n,$ then both $\dot{P}_m$ and $\dot{P}_n$ are δ_n -fine partitions; hence

$$\left|S\left(f;\dot{P}_n\right)-S\left(f;\dot{P}_m\right)\right|\le 1/n \qquad \text{for} \qquad m>n.$$

Consequently, the sequence $\left(S\left(f;\dot{P}_m\right)\right)_{m=1}^{\infty}$ is a Cauchy sequence in R, there-fore (see [B-S; p.82]) this sequence converges in ℝ and we let $A:=\lim_m S\left(f:\dot{P}_m\right).$ Passing to the limit as $m\to\infty$ in the above inequality, we have

$$\left|S\left(f;\dot{P}_n\right)-A\right|\le 1/n \qquad \text{for all} \qquad n\in R.$$

We now show that A is the integral of *f*. indeed, given $\varepsilon>0,$ let $K\in R$ with $k>2/\varepsilon.$ If $\dot{Q}$ is an arbitrary δ_k -fine partition, then

$$\left|S\left(f:\dot{Q}\right)-A\right|\le\left|S\left(f:\dot{Q}\right)-S\left(f;\dot{P}_K\right)\right|+\left|S\left(f;\dot{P}_K\right)-A\right|$$

$$\le 1/K+1/K<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, the function f is integrable to A.

3.3 THE INTEGRAL AS A FUNCTION OF INTERVALS

We will now show that if a function is integrable over an interval, then it is also integrable over any closed subinterval of that interval. In addition, the integral is "additive" in the sense of the next theorem.

Additivity Theorem. Let $f:[a,b]\to R$ and let $c\in(a,b)$. Then f is integrable on $[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both integrable. In this case we have

$$\int_a^b f=\int_a^c f+\int_c^b f.$$

Proof. $(\Leftarrow)$ Suppose that the restriction f_1 of f to the interval $I_1:=[a,c]$ and the restriction f_2 of f to $I_2:=[c,b]$ are integrable to A_1 and $A_{2,}$ respectively. Then, given $\varepsilon>0$, there is a gauge $\delta_\varepsilon^{'}$ on I_1 and a gauge $\delta_\varepsilon^{''}$ on I_2 such that if $\dot{P}_1$ is a $\delta_\varepsilon^{''}$-fine partition of I_2, then

$$\left|S(f_1;\dot{P}_1)-A_1\right|\le\frac{1}{2}\varepsilon \qquad \text{and}$$

$$\left|S(f_2;\dot{P}_2)-A_2\right|\le\frac{1}{2}\varepsilon.$$

We define a gauge δ_ε on $[a,b]$ by:

$$\delta_\varepsilon(t):=\begin{cases}\min\left\{\delta_\varepsilon^{'}(t),\frac{1}{2}(c-t)\right\} & if \quad t\in[a,c),\\ \min\left\{\delta_\varepsilon^{'}(c),\delta_\varepsilon^{''}(c)\right\} & if \quad t=c,\\ \min\left\{\delta_\varepsilon^{''}(t),\frac{1}{2}(t-c)\right\} & if \quad t\in(c,b].\end{cases}$$

Let $\dot{P}$ be a partition of $I:=[a,b]$ that is δ_ε – fine; then the point c must be a tag of at least one subinterval in $\dot{P}$, and we may use the right-left procedure to arrange that it is in two subintervals, and hence is a partition point of $\dot{P}$. Let $\dot{P}_1$ be the partition of I_1 consisting of the partition points $\dot{P} \cap I_1$, and let $\dot{P}_2$ be the partition of I_2 consisting of the partition points $\dot{P} \cap I_2$, so that

$$S(f;\dot{P}) - S(f_1;\dot{P}_1) + S(f_2;\dot{P}_2).$$

Since $\dot{P}$ is $\delta_\varepsilon^{'}$-fine and $\dot{P}_2$ is $\delta_\varepsilon^{''}$-fine, we conclude that

$$\left|S(f;\dot{P}) - (A_1 + A_2)\right| \le \left|S(f_1;\dot{P}_1) - A_1\right| + \left|S(f_2;\dot{P}_2) - A_2\right| \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then f is integrable on I and $(3.\theta)$ holds.

$(\Rightarrow)$ Conversely, suppose that f is integrable on I and, for each $\varepsilon > 0$, let η_ε be a gauge satisfying the Cauchy Criterion. As above, let f_1 denote the restriction of η_ε to I_1, and let $\dot{P}_1, \dot{Q}_1$ be partitions of I_1, that are $\eta_\varepsilon^{'}$-fine. By adjoining additional partition points and tags from I_2, we can extend $\dot{P}_1$ and $\dot{Q}_1$ to partitions $\dot{P}$ and $\dot{Q}$ of I that are η_ε-fine. If we use the same additional points and tags in I_2 for both $\dot{P}$ and $\dot{Q}$, it is easy to see that

$$S(f_1;\dot{P}_1) - S(f_1;\dot{Q}_1) = S(f;\dot{P}) - S(f;\dot{Q}).$$

But since $\dot{P}$ and $\dot{Q}$ are η_ε-fine, we conclude that $\left|S(f_1;\dot{P}_1) - S(f_1;\dot{Q}_1)\right| \le \varepsilon.$ Therefore, the Cauchy Critertion shows that the restriction f_1 of *f* to I_1 is integrable on I_1. In the same way, the restriction f_2 of *f* to I_2 is integrable. The equality $(3.\theta)$ now follows from the first part of the theorem

The next result is an important one; it will be seen later that the restriction of a generalized Riemann integrable function to an arbitrary set is not necessarily integrable.

Corollarly. If $f \in R^*([a,b])$ and $[c,d] \subseteq [a,b]$, then the restriction of f to $[c,d]$ is integrable.

Proof. Indeed, since f is integrable on $[a,b]$ and $c \in [a,b]$, then it follows from the theorem that the restriction of f to $[c,b]$ is integrable. But if $d \in [c,d]$, another application of the theorem shows that the restriction of f to $[c,d]$ is integrable. (We have used an obvious fact about restrictions of restrictions here; see Exercise 3.1.)

Corollary. If $f \in R^*([a,b])$ and if $a = c_0 < c_1 < ..... < c_n = b,$ then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are integrable and

(3.1) $$\int_a^b f = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f.$$

Proof. The assertion was proved for the case n = 2 in Theorem 3.7. The general case follows by using mathematical induction.

If $f \in R^*([a,b])$ and $\delta, \beta \in [a,b]$ with $\alpha < \beta,$ we have defined $\int_\alpha^\beta f$ to be the integral of the restriction of f to the subinterval $[\alpha, \beta]$. It is also convenient to define this integral for arbitrary values of $\alpha, \beta \in [a,b]$.

Definition. If $f \in R^*([a,b])$ and $\alpha, \beta \in [a,b], \alpha < \beta,$ we define

$$\int_\beta^\alpha f := -\int_\beta^\alpha f \qquad \text{and} \qquad \int_\alpha^\alpha f := 0.$$

Theorem. If $f \in R[a,b]$ and if α, β, γ are any numbers in $[a,b],$ then

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f,$$

In the sense that the existence of any two of these integrals implies the existence of the third integral and the equality $(3.k)$.

Proof. If any two of the numbers α, β, γ are equal, then $(3.k)$ holds. Thus we may suppose that all three of these numbers are distinct. For the sake of symmetry, we introduce the expression

$$L(\alpha, \beta, \gamma) := \int_\alpha^\beta f + \int_\alpha^\gamma f + \int_\gamma^\beta f.$$

It is clear that $(3.k)$ holds if and only if $L(\alpha, \beta, \gamma) = 0.$ Therefore, to establish the assertion, we need to show that L = 0 for all six permutations of the arguments α, β and γ.

We note that the Additivity theorem 3.7 implies that $L(\alpha,\beta,\gamma)=0.$ when $\alpha<\gamma<\beta.$ But it is easily seen that both $L(\beta,\gamma,\alpha)$ and $L(\gamma,\alpha,\beta)$ equal $L(\alpha,\beta,\gamma)$. Moreover, the numbers

$$L(\beta,\alpha,\gamma), \quad L(\alpha,\beta,\gamma) \quad \text{And} \quad L(\gamma,\beta,\alpha)$$

Are all equal to $-L(\alpha,\beta,\gamma).$ Therefore L vanishes for all possible configurations of these three points.

3.4 THE SQUEEZE THEOREM

The next result, a consequence of the Cauchy Criterion, is often useful in showing that a function is integrable.

Squeeze Theorem. A function f belongs to $R^*(I)$ if and only if for every $\varepsilon>0$ there exist functions φ_ε and ψ_ε in $R^*(I)$ with $\varphi_\varepsilon(x)\le f(x)\le\psi_\varepsilon(x)$ for all $x\in I,$ and such that

$$(3.\lambda) \qquad \int_I(\psi_\varepsilon-\varphi_\varepsilon)\le\varepsilon.$$

Proof. $(\Rightarrow)$ If $f\in R^*(I)$ and $\varepsilon>0$ is given, we can take $\varphi_\varepsilon:=\psi_\varepsilon:=f.$

$(\Leftarrow)$ Let $\varepsilon>0$ be given; then if $\varphi_\varepsilon\le f\le\psi_\varepsilon,$ it follows that for any tagged partition $\dot{P}$ of I we have

$$S(\varphi_\varepsilon;\dot{P})\le S(f;\dot{P})\le S(\psi_\varepsilon;\dot{P}).$$

Since $\varphi_\varepsilon\in R^*(I),$ there exists a gauge $\delta_\varepsilon^{'}>0$ on I such that if $\dot{P}<<\delta_\in^{'},$ then $\left|S(\varphi_\varepsilon;\dot{P})-\int_I\varphi_\varepsilon\right|\le\varepsilon,$ whence it follows that $\int_I\varphi_\varepsilon-\varepsilon\le S(\varphi_\varepsilon;\dot{P}).$ Similarly there exists a gauge $\delta_\in^{''}>0$ such that if $\dot{P}<<\delta_\in^{''},$ then $S(\psi_\varepsilon;\dot{P})\le\int_I\psi_\varepsilon+\varepsilon.$ Now let $\delta_\varepsilon:=\min\{\delta_\varepsilon^{'},\delta_\varepsilon^{''}\},$ so that if $\dot{P}<<\delta_\varepsilon,$ then

$$\int_I\varphi_\varepsilon-\varepsilon\le S(f;\dot{P})\le\int_I\psi_\varepsilon+\varepsilon,$$

and if $\dot{Q}<<\delta_\varepsilon,$ then

$$-\int_I \psi_\varepsilon - \varepsilon \le -S\left(f;\dot{Q}\right) \le -\int_I \varphi_\varepsilon + \varepsilon.$$

Adding these inequalities, we obtain

$$-\int_I (\psi_\varepsilon - \varphi_\varepsilon) - 2\varepsilon \le S\left(f;\dot{P}\right) - S\left(f;\dot{Q}\right) \le \int_I (\psi_\varepsilon - \varphi_\varepsilon) + 2\varepsilon,$$

Whence we conclude from $(3.\lambda)$ that

$$\left|S\left(f;\dot{P}\right) - S\left(f;\dot{Q}\right)\right| \le \int_I (\psi_\varepsilon - \varphi_\varepsilon) + 2\varepsilon \le 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, *f* satisfies the Cauchy Criterion. Therefore *f* is integrable on I.

Step Functions

We now establish the integrability of certain important classes of functions.

First, we will discuss step functions, and then turn to more complicated

classes.

Definition. A function $s : I \to \mathbb{R}$ is said to be a step function on $I := [a,b]$ if there exists a partition $\{[c_i - 1, c_i]\}_{i=1}^{n}$ of *I* and real numbers $\{\alpha_i\}_{i=1}^{n}$ such that

$$(3.\mu) \qquad s(x) = \alpha_i \text{ for } \quad x \in (c_{i-1}, c_i), i = 1, \ldots\ldots, n.$$

Remark. The step function *s* also has values at the partition points which may differ from the values α_i. For the purposes of integration, these values at c_i are totally unimportant, as is *seen* in Exercises 1.R or 3.C.

Theorem. Every step function on $I := [a,b]$ is integrable. In fact, if s is the step function given by $(3.\mu)$, then

$$(3.\upsilon) \qquad \int_a^b s = \sum_{i=1}^{n} \alpha_i (c_i - c_{i-1}).$$

Proof: Define s_i on I by $s_i(x) := \alpha_i$ for $x \in (c_{i-1}, c_i)$ and $s_i(x) := 0$ else where one I. We have seen in Example 2.2.(b)

Notes

and Exercise 2.J hat $s_i \in R^*(I)$ with integral $\alpha_i(c_i - c_{i-1})$. Now apply Theorem 3.1 and induction. Q.E.D.

Regulated Functions We now introduce an important (and a quite inclusive) class of functions on the interval $I := [a,b]$ that will be seen to be integrable. It will be shown that continuous and monotone functions are contained in this class.

Definition: A function $f : I \to \mathbb{R}$ is said to be regulated on $I := [a,b]$ if for every $\varepsilon > 0$ there exists a step function $s_\varepsilon : I \to \mathbb{R}$ such that

$$(3.\xi) \qquad |f(x) - s_\varepsilon(x)| \le \varepsilon \quad \text{for all} \quad x \in I.$$

Remark. By letting $\varepsilon = 1/n \, (n \in \mathbb{N})$, it is clear that a function f is regulated if and only if there is a sequence $(s_n)_{n=1}^{\infty}$ of step functions on $I \to \mathbb{R}$ that converges uniformly to f on I (see [B-X; p.229] or Definition 8.2 below).

Integrability of regulated functions. If $f : I \to \mathbb{R}$ is a regulated function on $I := [a,b]$ then $f \in \mathcal{R}^*(I)$.

Proof. Given $\varepsilon > 0$. Let $s_\varepsilon : I \to \mathbb{R}$ be a step function such that $(3.\xi)$ holds. Therefore, we have

$$s_\varepsilon(x) - \varepsilon \le f(x) \le s_\varepsilon(x) + \varepsilon \qquad \text{for} \quad x \in [a,b].$$

If we let $\varphi_\varepsilon(x) := s_\varepsilon(x) - \varepsilon$ and $\psi_\varepsilon(x) := s_\varepsilon(x) + \varepsilon$ for $x \in I$, then the step functions φ_ε and ψ_ε are integrable and $\varphi_\varepsilon(x) \le f(x) \le \psi_\varepsilon(x)$ for $x \in I$. Moreover, since

$$\int_a^b (\psi_\varepsilon - \varphi_\varepsilon) = \int_a^b 2\varepsilon = 2(b-a)\varepsilon,$$

It follows from the Squeeze theorem 3.12 that $f \in R^*(I)$.

The following characterization of regulated functions will be useful.

3.5 CHARACTERIZATION OF REGULATED FUNCTIONS.

A function $f : I \to R$ is a regulated function if and only if it has all of its one-sided limits at every point of the interval $I := [a,b]$.

Proof. $(\Rightarrow)$ First we note that every step function has one-sided limits at each point. To prove that a regulated function f has the same property, let $c \in [a,b)$; we will prove that f has a right hand limit at c. to do so, let $\varepsilon > 0$ be given and let $s_\varepsilon : I \to R$ be a step function such that $(3.\xi)$ holds. Since s_ε is a step function and $\lim_{x \to c+} s_\varepsilon(x)$ exists, there exists $\delta_\varepsilon(c) > 0$ such that if $x, y \in (c, c+\delta_\varepsilon(c))$, then $s_\varepsilon(x) = s_\varepsilon(y)$. Therefore, if $x, y \in (c, c+\delta_\varepsilon(c))$, then

$$|f(x) - f(y)| \le |f(x) - s_\varepsilon(x)| + |s_\varepsilon(x) - s_\varepsilon(y)| + |s_\varepsilon(y) - f(y)|$$

$$\le \varepsilon + 0 + \varepsilon = 2\varepsilon.$$

But since $\varepsilon > 0$ is arbitrary, the Cauchy criterion implies that the right hand limit $\lim_{x \to c+} f(x)$ exists.

The existence of left hand limits at $c \in (a,b]$ is proved in the same way.

$(\Leftarrow)$ Suppose f has all one-sided limits guarantees that given $\varepsilon > 0$, there is a gauge δ_ε on I such that if $t \in I$ and y_1, y_2 are both in $[t - \delta_\varepsilon(t), t)$, or are both in $(t, t + \delta_\varepsilon(t)]$, then $|f(y_1) - f(y_2)| \le \varepsilon$.

Now let $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ be a $\delta_\varepsilon -$fine partition of I. We define $s_\varepsilon(z) := f(z)$ if z is one of the numbers

$$a = x_0 \le t_1 \le ..... \le x_{i-1} \le t_i \le x_i \le ...... \le t_n \le x_n = b.$$

On the interval $(x_{i-1}, t_i) \subseteq [t_i - \delta_\varepsilon(t_i), t_i),$ we define $s_\varepsilon(x) := f\left(\frac{1}{2}(x_{i-1}+t_i)\right)$ so that

$$\left|f(x) - s_\varepsilon(x)\right| = \left|f(x) - f\left(\frac{1}{2}(x_{i-1}+t_i)\right)\right| \le \varepsilon.$$

Similarly, on the interval $(t_i, x_i) \subseteq (t_i, t_i + \delta_\varepsilon(t_i)],$ we define $s_\varepsilon(x) := f\left(\frac{1}{2}(t_i + x_i)\right),$ so that

$$\left|f(x) - s_\varepsilon(x)\right| = \left|f(x) - f\left(\frac{1}{2}(t_i + x_i)\right)\right| \le \varepsilon.$$

Hence the step function s_ε satisfies $\left|f(x) - s_\varepsilon(x)\right| \le \varepsilon$ for all $x \in I.$ But since $\varepsilon > 0$ is arbitrary, we conclude that f is a regulated function.

Integrability of continuous functions : If $f : I \to R$ is continuous on $I := [a, b]$ then $f \in R^*(I).$

Proof. Since a continuous function on I has a limit at every point of I, theorem implies that a continuous function is a regulated function. Hence, by previous theorem a continuous function is integrable on I.

We recall that a function $f : I \to R$ is said to be increasing (or non-decreasing) on I if $x, y \in I, x \le y,$ imply that $f(x) \le f(y).$ Similarly, f is said to be decreasing (or nonincreasing) on I if $x, y \in I, x \le y,$ imply that $f(x) \le f(y).$ A function is said to be monotone on I if it is either increasing on I or decreasing on I.

Integrability of monotone functions. If $f : I \to R$ is monotone on $I := [a, b],$ then f is regulated and $f \in R^*(I).$

Proof. It is known (see [B-S; p.149]) that a monotone function on I has one-sided limits at every point of I. It follows from theorem 3.17 that a monotone function is a regulated function. Therefore, by Theorem 3.16, a monotone function is integrable on I.

The next result about regulated functions will be used in Section 4.

Theorem: The set of points of discontinuity of a regulated function $f: I \to R$ is a countable subset of $I := [a,b]$.

Proof. For each $n \in R,$ let $s_n : I \to R$ be a step function such that

$$\left| f(x) - s_n(x) \right| \le 1/n \qquad \text{for} \qquad x \in I.$$

Since the step function s_n has finite set D_n of points of discontinuity, the set $D := \bigcup_{n=1}^{\infty} D_n$ is a countable set in I. We will show that if $c \in I - D,$ then c is a point of continuity of f. Indeed, given $\varepsilon > 0,$ choose $N > 1/\varepsilon,$ so that $\left| f(x) - s_N(x) \right| \le 1/N < \varepsilon$ for all $x \in I.$ Since s_N is continuous at c, there exists $\gamma > 0$ such that if $|x - c| < \gamma, x \in I,$ then $\left| s_N(X) - s_N(c) \right| \le \varepsilon.$ Combining these estimates, we conclude that if $|x - c| < \gamma, x \in I,$ then

$$\left| f(x) - f(c) \right| \le \left| f(x) - s_N(x) \right| + \left| s_N(x) - s_N(c) \right| + \left| s_N(c) - f(c) \right|$$

$$\le \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then f is continuous at $c \in I - D.$

It is easy to see that the product of two integrable functions is not necessarily integrable. However, the following partial result is sometimes useful; a stronger theorem will be given in 10.12.

Theorem: Let $f \in R^*([a,b])$ be bounded below and let g be a regulated function on $[a,b].$ Then the product f. g belongs to $R^*([a,b]).$

Proof. It is evidently enough to consider the case where $f(x) \ge 0$ for $x \in I := [a,b].$ It is also clear that if s is a step function, then f .s is integrable. Let $A > \int_a^b f \ge 0$ and let $\varepsilon > 0.$ If g is a regulated function, let s_ε be a step function on I such that $\left| g(x) - s_\varepsilon(x) \right| \le \varepsilon / 2A$ for all $x \in I.$ If we define $\varphi_\varepsilon(x) := f(x)\left(s_\varepsilon(x) - \varepsilon/2A\right)$ and $\psi_\varepsilon(x) := f(x)\left(s_\varepsilon(x) + \varepsilon/2A\right)$ for all $x \in I,$ then $\varphi_\varepsilon, \psi_\varepsilon \in R^*(I)$ and it follows that

Notes

$$\varphi_\varepsilon(x) \le f(x).g(x) \le \psi_\varepsilon(x) \qquad \text{for all} \qquad x \in I,$$

and that

$$\int_I (\psi_\varepsilon - \varphi_\varepsilon) = (\varepsilon / A)\int_I f \le \varepsilon.$$

Therefore the Squeeze theorem 3.12 implies that $f.g \in R^*(I)$.

Translations

We will close this section with a theorem showing that the translation (either additive or multiplicative) of an R*-integrable function is R*-integrable. These results are special instances of the Substitution Theorem that will be discussed in Section 13.

Let $I := [a,b]$ be a compact interval in R and let $r \in R$. We define the r-additive translate of I to be the interval $I_r := [a+r, b+r]$, and the r-additive translate of f to be the function $f_r(y) := f(y-r)$ for all $y \in I_r$. Similarly, if $r > 0$, we define the r-multiplicative translate of I to be the interval $I_{(r)} := [ar, br]$, and the r-multiplicative translate of f to be the function $f_{(r)}(z) := f(z/r)$ for all $z \in I_{(r)}$. (If $r < 0$, then the multiplicative translates can also be defined, except, that the order of the points in the interval is reversed.)

Theorem. (a) If $f \in R^*(I)$, then $f_r \in R^*(I_r)$ and

$$(3.o) \qquad \int_{I_r} f_r = \int_I f.$$

(b) If $f \in R^*(I)$ and $r > 0$, then $f_{(r)} \in R^*\left(I_{(r)}\right)$ and

$$(3.\pi) \qquad \int_{I_{(r)}} f_{(r)} = r\int_I f.$$

Proof. (a) Given $\varepsilon > 0$, let δ_ε be a gauge on I such that if $\dot{P}_1$ is any $\delta_\varepsilon -$ fine partition of I, then $\left|S(f;\dot{P}_1) - f_{If}\right| \le \varepsilon$. Now define $\eta_\varepsilon(s) := \delta_\varepsilon(s-r)$ for all $s \in I_r$ so that η_ε is a gauge on the interval I_r.

Suppose that $\dot{Q} := \{([y_{i-1}, y_i], s_i)\}_{i=1}^n$ is an η_ε -fine partition of I_r, whence

$$s_i - \eta_\varepsilon(s_i) \le y_{i-1} \le s_i \le y_i \le s_i + \eta_\varepsilon(s_i).$$

If we let $x_i := y_i - r$ and $t_i := s_i - r,$ then $\eta_\varepsilon(s_i) = \delta_\varepsilon(s_i - r) = \delta_\varepsilon(t_i),$ so that

$$t_i - \delta_\varepsilon(t_i) \le x_{i-1} \le t_i \le x_i \le t_i + \delta_\varepsilon(t_i),$$

Whence $\dot{P} := \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is a $\delta_\varepsilon -$fine partition of I. Since we readily see that $S(f_r;\dot{Q}) = S(f;\dot{P}),$ we infer that

$$\left|S(f_r;\dot{Q}) - \int_I f\right| = \left|S(f;\dot{P}) - \int_I f\right| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f_r \in R^*(I_r)$ and that $(3.o)$ holds.

(b) Given $\varepsilon > 0$, let δ_ε be a gauge on I such that if $\dot{P}_1$ is any δ_ε-fine partition of I, then $\left|S(f;\dot{P}_1) - \int_I f\right| \le \varepsilon / r.$ Now defined $\xi_\varepsilon(z) := r\delta_\varepsilon(z/r)$ for $z \in I_{(r)}$ so that ξ_ε is a gauge on $I_{(r)}.$

Suppose that $\dot{u} := \{([z_{i-1}, z_i], u_i)\}_{i=1}^{n}$ is a ξ_ε-fine partition of $I_{(r)}$, whence

$$u_i - \xi_\varepsilon(u_i) \le z_{i-1} \le u_i \le z_i \le u_i + \xi_\varepsilon(u_i).$$

If we let $x_i := z_i / r$ and $t_i := u_i / r,$ then $\xi_\varepsilon(u_i) = \xi_\varepsilon(rt_i) = r\delta_\varepsilon(t_i),$ so that

$$t_i - \delta_\varepsilon(t_i) \le x_{i-1} \le t_i \le x_i < t_i + \delta_\varepsilon(t_i),$$

Whence $\dot{P} := \{[x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ is a $\delta_\varepsilon -$fine partition of I. Since we readily see that $S(f_{(r)};\dot{u}) = rS(f;\dot{P}),$ we infer that

$$\left|S(f_{(r)};\dot{u}) = r\int_I f\right| = r.\left|S(f;\dot{P}) - \int_I f\right| \le r.(\varepsilon / r) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $f_{(r)} R^*(I_{(r)})$ and $(3.\pi)$ holds. Q.E.D.

Exercises

3.A Write out the details of the proof of Theorem 3.1(b).

3.B Use induction to show that if $f_1,....,f_n$ are in $R^*(I)$ and if $c_1,....,c_n$ are in R, then the linear combination $f := \sum_{k=1}^{n} c_k f_k$ belongs to $R^*(I)$ and $\int_I f = \sum_{k=1}^{n} c_k \int_I f_k$.

3.C Suppose that $f \in R^*(I)$, that $g : I \to R$, and that $g(x) = f(x)$ a.e. on I. Show that $g \in R^*(I)$ and that $\int_I g = \int_I f$. [Hint: Consider $h := g - f$.]

3.D Suppose that $f, g \in R^*(I)$ and that $f(x) \le g(x)$ a.e. on I. Show that $\int_I f \le \int_I g$.

3.E If $f, g \in R^*(I)$ and $|f(x)| \le g(x)$ a.e. on I, show that $\left|\int_I f\right| \le \int_I g$.

3.F Suppose that $f, g : I \to R$, that $g \in R^*(I)$ with $\int_I g = 0$, and that $|f(x)| \le g(x)$ for all $x \in I$. Show that f and $|f|$ are integrable and that $\int_I f = 0 = \int_I |f|$.

3.G Show that the conclusion in Exercise 3.F remains true under the hypothesis that $|f(x)| \le g(x)$ a.e. on I.

3.H Write out a proof of the quality $S(f;\dot{P}) = S(f_1;\dot{P}_1) + S(f_2;\dot{P}_2)$ used in the proof of Theorem 3.7. Also, establish the relation $S(f_1;\dot{P}_1) - S(f_1;\dot{Q}_1) = S(f;\dot{P}) - S(f;\dot{Q})$ that was used in the second part of that proof.

3.I Let $f : [a,b] \to R$ and let $[c,d] \subseteq [a,b]$. Let f_1 be the restriction $f \| [c,b]$ of f to $[c,b]$, and let f_2 be the restriction $f_1 \| [c,d]$. Show that $f_2 = f \| [c,d]$.

3.J Let $f \in R^*([a,b])$ and let $c \in (a,b)$. If $g(x) := 0$ for $x \in [a,c)$ and if $g(x) := f(x)$ for $x \in [c,b]$, show that $g \in R^*([a,b])$ and that $\int_a^b g = \int_c^b f$.

3.K Suppose that (f_n) is a sequence in $R^*(I)$ such that $0 \le f_n(x) \le f(x)$ for all $x \in I$, and $\int_I f_n \ge n$ for all $n \in R$. Show that $f \notin R^*(I)$.

3.L Show that the function $f(x) := 1/x$ for $x \in (0,1]$ and $f(0) := 0$ is not in $f(0) := 0$ [Hint: Construct step functions f_n with $f_n \le f$.]

Check your progress

1. Prove: If $f \in R^*([a,b])$ and $[c,d] \subseteq [a,b]$, then the restriction of f to $[c,d]$ is integrable.

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2. Prove: If $f \in R^*([a,b])$ and if $a = c_0 < c_1 < \ldots.. < c_n = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are integrable and

$$\int_a^b f = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f.$$

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3. Prove: If $f \in R[a,b]$ and if α, β, γ are any numbers in $[a,b]$, then

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f,$$

Prove: A function $f : I \to R$ is a regulated function if and only if it has all of its one-sided limits at every point of the interval $I := [a,b]$.

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3.6 LET US SUM UP

Notes

1. If f and g are integrable on I to R, then their sum $f+g$ is also integrable on I and

$(3.\alpha)$ $$\int_I (f+g) = \int_I f + \int_I g.$$

(b) If f is integrable on I and $c \in R,$ then *cf* is integrable on I and

$(3.\alpha)$ $$\int_I cf = c\int_I f.$$

2. If $f \in R^*(I)$ and $f(x) \geq 0$ for all $x \in I,$ then

$(3.\gamma)$ $$\int_I f \geq 0.$$

3. Let $f:[a,b] \to R$ and let $c \in (a,b).$ Then f is integrable on $[a,b]$ if and only if its restrictions to $[a,c]$ and $[c,b]$ are both integrable. In this case we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

4. If $f \in R^*([a,b])$ and $[c,d] \subseteq [a,b],$ then the restriction of f to $[c,d]$ is integrable.

5. If $f \in R^*([a,b])$ and if $a = c_0 < c_1 < ..... < c_n = b,$ then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are integrable and

$$\int_a^b f = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f.$$

6. If $f \in R[a,b]$ and if α, β, γ are any numbers in $[a,b],$ then

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f,$$

7. A function $f: I \to R$ is a regulated function if and only if it has all of its one-sided limits at every point of the interval $I := [a,b].$

8. Let $f \in R^*([a,b])$ be bounded below and let g be a regulated function on $[a,b].$ Then the product f. g belongs to $R^*([a,b]).$

3.7 KEY WORDS

Regulated function

Discontinuity of a regulated function

The integral as a Function of Intervals

The Squeeze Theorem

Characterization of regulated functions

3.8 QUESTIONS FOR REVIEW

1. Discuss about The integral as a Function of Intervals

2. Prove the Squeeze Theorem

3. Explain about Characterization of regulated functions

3.11 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

3.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 3.4

2. See section 3.4

3. See section 3.6

UNIT -4 THE FUNDAMENTAL THEOREMS OF CALCULUS

STRUCTURE

4.0 Objective

4.1 Introduction

4.2 Definitions related to theorems of calculus

4.3 Remarks

4.4 Examples

4.5 Fundamental theorems of Integrals I

4.6 Fundamental theorems of Integrals II

4.7 Let us sum up

4.8 Key words

4.9 Questions for review

4.10 Suggestive readings and references

4.11 Answers to check your progress

4.0 OBJECTIVE

In this unit we will learn and understand about theorems related fundamental calculus, remarks, examples and Fundamental theorems.

4.1 INTRODUCTION

There are two aspects to what is traditionally called the "Fundamental Theorem of Calculus": one part is concerned with the integration of derivatives, and the other part with the differentiation of integrals. Both aspects will be discussed in this section.

The reader first encountered the Fundamental Theorem in a course on calculus and learned to evaluate the integral of a function f known to be R-integrable by finding a function F on $[a,b]$ with $F^{'}(x)=f(x)$ for all $x\in[a,b]$, and then evaluating $F(b)-F(a)$. This method — which is essentially due to Newton and Leibniz — is the one used to evaluate virtually all elementary integrals (although we will later obtain some convergence theorems that can be also useful). In any case, we seldom evaluate integrals directly using the definition, or by calculating Riemann sums. The reader may imagine that, since the usual Newton-Leibniz formula enables one to evaluate R-integrals, then a more complicated method will be required to evaluate generalized Riemann integrals. In fact, as we will see, the rule for evaluating R* - integrals is the same as for R-integrals.

Moreover, since the derivative of a function is always generalized Riemann integrable (but not always R-integrable or even L-integrable), the situation is actually *simpler* for the R*-integral than for the R-integral.

In *this* section, we will employ a "spiral" approach and establish weaker or more elementary results first, and then establish stronger results that require slightly more complicated proofs.

Primitives and Indefinite Integrals

Before we state our main theorems, it is convenient to introduce some terminology. In the following material, it will often be convenient to consider three types of *exceptional sets:* finite sets, countable sets, and null sets. In doing so, we will use the prefixes $f-, c-,$ and $a-$ (for almost everywhere).

4.2 DEFINITIONS OF FUNDAMENTAL CALCULUS

Let $I:=[a,b]\subset R$ and let $F, f: I\to R.$

Notes

(a) We say that F is a primitive (or an antiderivative) of f *on* I if the derivative $F'(x)$ exists and $F'(x) = f(x)$ *for all* $x \in I$.

(b) We say that F is an a-primitive [respectively, **c-primitive, f-**primitive] of f on I if F is continuous on I, and there exists a null [respectively, countable, finite] set E of points $x \in I$ where either $F'(x)$ does not exist, or does not equal $f(x)$. The set E is called the exceptional set of f.

(c) If $f \in R^*(I)$ and $u \in I$, then the function $F_u : I \to R$ defined by $(4.\alpha)$

$$F_u(x) := \int_u^x f$$

is called the **indefinite integral of *f* with base point** u. If the base point is the left endpoint (or is well understood) We usually do not write the subscript. Any function that differs from F_a. by a constant is called an indefinite integral of *f*.

WARNING. Some authors use the words "antiderivative", "primitive", and "indefinite integral" as synonyms, or make distinctions that are slightly different from the ones used here.

4.3 REMARKS

(a) In Definition 4.1(a) we did not need to assume that F is continuous on I, since the existence of its derivative on I guarantees this to be the case. But, in 4.1(b), it is important to assume that F is continuous on I.

(b) For c *in* an exceptional set E the derivative $F'(c)$ *may* not exist, or $F'(c)$ may exist but not equal $f(c)$. In fact, sometimes the function f is not even defined at certain points in E; in this case we extend f to be equal to 0 at such points.

(c) If *f* is integrable on $[a,b]$, then it follows from Corollary 3.8 and Definition 3.10 that the integral in $(4.\alpha)$ is defined, so Definition 4.1(c) makes sense. Note that we always have $F_u(u)=0.$

d) If F_u is the indefinite integral of *f* with base point *u*, then since $F_u(x)=F_a(x)-\int_a^u f$ for $x\in I$, it follows that F_u *is* an indefinite integral of *f*.

4.4 EXAMPLES

(a) If $f(x):=x^n$ for $x\in[a,b], n\in R,$ then $F(x):=x^{n+1}/(n+1)$ is a primitive of *f* on any interval $[a,b]\subset R.$ It will be seen later that *F* is also the indefinite integral of *f* with base point 0.

(b) If $g(x):=1/\sqrt{x}$ for $x\in(0,1]$, then *g* is not defined at *x* = 0 so we define $g(0):=0$. Further, *g* is not bounded on $[0,1]$, so its R-integral does *not exist*. Nevertheless, we will see in Example 4.6 that *g* is R*-integrable. In any case, it is true that the function $G(x):=2\sqrt{x}$ for $x\in[0,1]$ is an f-primitive of *g* on [0,1] with the (finite) exceptional set *E* = {0}. It will be seen later that *G* is also the indefinite integral of *g* with base point 0.

(c) If sgn is the signum function on R defined by

$$\operatorname{sgn}(x):=\begin{cases}-1 & if \quad x<0,\\ 0 & if \quad x=0,\\ +1 & if \quad x>0,\end{cases}$$

then sgn is integrable over any interval $[a,b]\subset R,$ since its restriction to this interval is a step function. It is easy to see that the function $H(x):=|x|$ *is* an f-primitive of sgn on any interval $[a,b]$ with exceptional set $E=\{0\}$.

It is an exercise to show that H is the indefinite integral of sgn with base point 0.

(d) Let f be the Dirichlet function introduced in Example 2.3(a), which is everywhere discontinuous, yet R^*- integrable on $I:=[0,1]$. If we let $F(x):=0$ for all $x \in I$, then F *is* a c-primitive of f on I since $F'(x)=0=f(x)$ for all irrational numbers $x \in I$. Here the exceptional set E is the set of all rational numbers in I, which is a countable set. Also, it is easily seen that the zero function F is the indefinite integral of f with base point 0.

The Straddle Lemma

We will discuss primitives and indefinite integrals later in this section. In order to prove the Fundamental Theorem I, we need a lemma that is a direct consequence of the definition of the derivative. The reader should observe that the points u, v "straddle" the point t; that explains the name given the lemma.

Straddle Lemma.

Let $F: I \to R$ be *differentiable at* a point $t \in I$. *Given* $\varepsilon > 0$ there *exists* $\delta_\varepsilon(t) > 0$ *such that if* $u, v \in I$ *satisfy*

$t-\delta_\varepsilon(t) \le u \le t \le v \le t+\delta_\varepsilon(t)$, then we have

$$|F(v)-F(u)-F'(t)(v-u)| \le \varepsilon(v-u). \qquad (4.\beta)$$

Proof. By definition of the derivative $F'(t)$ at the point $t \in I$, given $\varepsilon > 0$ there exists $\delta_\varepsilon(t) > 0$ such that if

$0 < |z-t| \le \delta_\varepsilon(t), z \in I,$ then

$$\left|\frac{F(z)-F(t)}{z-t} - F'(t)\right| \le \varepsilon,$$

Hence it follows that

$$|F(z)-F(t)-F'(t)(z-t)| \le \varepsilon|z-t|$$

For all $z \in I$ such that $|z - I| \le \delta_\varepsilon(t)$.

In particular, if we pick $u \le t$ and $v \ge t$ in this $\delta_{Ge}(t)-$interval around t and note that $v - t \ge 0$ and $t - u \ge 0,$ then on subtracting and adding the term $F(t) - F'(t)t,$ we have

$$|F(v) - F(u) - F'(t)(v-u)|$$

$$= |[F(v) - F(t) - F'(t)(v-t)] - [F(u) - F(t) - F'(t)(u-t)]|$$

$$\le |F(v) - F(t) - F'(t)(v-t)| + |F(u) - F(t) - F'(t)(u-t)|$$

$$\le \varepsilon(v-t) + \varepsilon(t-u) = \varepsilon(v-u).$$

Thus inequality $(4.\beta)$ is proved.

Integrating Derivatives

We now establish the first of several *versions* of the Fundamental Theorem that guarantees that the derivative of any function on an interval $I := [a,b]$ is always R*-integrable, without imposing further hypotheses on this derivative. It was in order to obtain this result that Denjoy and Perron developed their theories of integration. (Stronger results will be obtained in Theorems 4.7 and 5.12.)

4.5. FUNDAMENTAL THEOREM I.

If $f : [a,b] \to R$ has a primitive F on $[a,b]$, then $f \in R^*([a,b])$ and

$$(4.\gamma) \qquad \int_a^b f = F(b) - F(a).$$

Proof. Given $\varepsilon > 0,$ *let* the gauge δ_ε, be as in the Straddle Lemma, and let $\dot{P} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be a δ_ε-fine partition of $[a,b]$. Since x_{i-1} and x_i straddle the tag t_i, it follows from $(4.\beta)$ that

$$(4.\delta) \qquad |F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \le \varepsilon(x_i - x_{i-1}).$$

Notes

We wish to estimate the quantity $F(b)-F(a)-S(f;\dot{P})$. To do so we make use of the telescoping sum $F(b)-F(a)=\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]$ to obtain

$$F(b)-F(a)-S(f;\dot{P})=\sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})-f(t_i)(x_i-x_{i-1})\right].$$

We use the Triangle inequality, to obtain the inequality

$$\left|F(b)-F(a)-S(f;\dot{P})\right|\le\sum_{i=1}^{n}\left|F(x_i)-F(x_{i-1})-f(t_i)(x_i-x_{i-1})\right|.$$

It follows from $(4.\delta)$ that the last term is dominated by the telescoping sum

$$\sum_{i=1}^{n}\varepsilon(x_i-x_{i-1})=\varepsilon(b-a).$$

Since $\varepsilon>0$ is arbitrary, we conclude that $f\in R^*([a,b])$ with integral equal to $F(b)-F(a)$.

***We* can restate the Fundamental Theorem 4.6** *as:* If $F:[a,b]\to R$ is differentiable at every point of $[a,b]$, then $F'\in R^*([a,b])$ and we have $\int_a^b F'=F(b)-F(a)$.

It is an exercise to show that if F is a primitive on $[a,b]$ of a function *f* and $u\in[a,b]$, then $F-F(u)$ is the indefinite integral of *f* with base point *u*.

In the next example, we show that the proof of the Fundamental Theorem 4.6 can be modified to permit one point of nondifferentiability. These ideas will enable us to strengthen 4.5.

Example. Let $g(x):=1/\sqrt{x}$ for $x\in(0,1]$ and $g(0):=0$, so that g is not bounded on $[0,1]$. If $G(x):=2\sqrt{x}$ for $x\in[0,1]$, then G is continuous on $[0,1]$ and $G'(x)=g(x)$ for all $x\in(0,1]$, but $G'(0)$

does not exist. Hence G is an f-primitive (but not a primitive) of g on $[0,1]$ with exceptional set $E=\{0\}$.

As in the proof of the Straddle Lemma, if $t\in(0,1]$ and $\varepsilon>0$, let $\delta_\varepsilon(t)>0$ be such that the inequality $(4.\beta)$ holds for G. We define $\delta_\varepsilon(0):=\varepsilon^2/4$ so that if $0\le v\le\delta_\varepsilon(0)$, then $G(v)-G(0)=2\sqrt{v}\le\varepsilon$. Now let $\dot{P}:=\{([x_{i-1},x_i],t_i)\}_{i=1}^n$ be a tagged partition of I that is $\delta_\varepsilon-$fine. If all of the tags belong to $(0,1]$, then the proof of the Fundamental Theorem 4.6 applies without change. However, if the first tag $t_i=0$, then the first term in the Riemann sum $S(g;\dot{P})$ is equal to $g(0)(x_1-x_0)=0$; moreover, we have

$$G(x_1)-G(x_0)-g(0)(x_i-x_0)=G(x_1)=2\sqrt{x_1}\le\varepsilon.$$

If we apply the argument given in the proof of the Fundamental Theorem 4.6 to the remaining terms, we obtain

$$\left|\sum_{i=2}^{n}\left[G(x_1)-G(x_{i-1})-g(t_i)(x_i-x_{i-1})\right]\right|\le\varepsilon(x_n-x_1)\le\varepsilon.$$

Therefore, on adding these terms we have

$$\left|G(1)-G(0)-S(g;\dot{P})\right|\le\varepsilon+\varepsilon=2\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude that g is R*-integrable on $[0,1]$ and that $\int_0^1 g=G(1)-G(0)=2.$ we may write this in the form $\int_0^1(1/\sqrt{x})dx=2,$ with the understanding that the integrand is given the value 0 at the point where it is not defined.

Clearly, the argument in Example 4.6 can be extended to any integrand *f* such that there exists a continuous function *F* on $[a,b]$ such that $f(x)=F'(x)$ for all but a finite number of points. We now show that, in fact, **a** *countable* number of exceptional points is permitted. This yields a significant improvement over Theorem 4.6.

Fundamental Theorem I*. If $f:[a,b]\to R$ has a c-primitive F on $[a,b]$, then $f\in R^*([a,b])$ and

$$(4.\varepsilon) \qquad \int_a^b f = F(b)-F(a).$$

Proof. Let $E=\{c_k\}_{k=1}^{\infty}$ be the exceptional set for the c-primitive *F*. Since *E* is countable, it is a null set. In view of Exercise 3.C, we may suppose that $f(c_k)=0.$

We now define a gauge on $I:=[a,b]$. If $\varepsilon>0$ and $t\in I-E$, be as in the Straddle Lemma. If $t\in E$, then $t=c_k$ for some $k\in\mathbb{N}$; from the continuity of *F* at c_k, we choose $\delta_\varepsilon(c_k)>0$ such that $\left|F(z)-F(c_k)\right|\le\varepsilon/2^{k+2}$

for all $z\in I$ that satisfy $|z-c_k|\le\delta_\varepsilon(c_k)$.. Thus, a gauge δ_ε is defined on *I*.

Now let $\dot{P}:=\{([x_{i-1}x_i],t_i)\}_{i=1}^{n}$ be a δ_ε-fine partition of *I*. If none of the tags belongs to *E*, then the proof given in Theorem 4.5 applies without change. However, if $c_k\in E$ is the tag of a subinterval $[x_{i-1},x_i]$, then

$$\left|F(x_i)-F(x_{i-1})-f(c_k)(x_i-x_{i-1})\right|$$

$$\le\left|F(x_i)-F(c_k)\right|+\left|F(c_k)-F(x_{i-1})\right|+\left|f(c_k)(x_i-x_{i-1})\right|$$

$$\le\varepsilon/2^{k+2}+\varepsilon/2^{k+2}+0=\varepsilon/2^{k+1}.$$

Now each point of *E* can be the tag of at most two subintervals in $\dot{P}$; therefore the sum of the terms with $t_i\in E$ satisfies

$$\sum_{t_i\in E}\left|F(x_i)-F(x_{i-1})-f(t_i)(x_i-x_{i-1})\right|\le\sum_{k=1}^{\infty}\varepsilon/2^k=\varepsilon.$$

Also, by the Straddle Lemma, the sum of the terms with $t_i\notin E$ satisfies

$$\sum_{t_i \notin E} \left| F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1}) \right| \le \varepsilon \sum_{t_i \notin E} (x_i - x_{i-1}) \le \varepsilon (b-a).$$

Consequently, when $\dot{P} << \delta_{\varepsilon}$, , we have

$$\left| F(b) - F(a) - S(f; \dot{P}) \right| \le \varepsilon (1 + b - a).$$

Since $\varepsilon > 0$ is arbitrary, then $f \in R^*(I)$ with integral $F(b) - F(a)$.

Theorem 4.7 can be stated: If *F is* a c-primitive *of* f on $[a,b]$, then $f \in R^*([a,b])$ and F is an indefinite integral of *f*.

Differentiating Integrals

We now turn to the part of the Fundamental Theorem that discusses the differentiation of an indefinite integral. Here the situation is not as definitive as in the first part of the Fundamental Theorem.

However, it is true that an indefinite integral of an R*-integrable function is continuous on I. (In Exercise 4.K one shows that this is true for a bounded function, and the general case will be proved in Section 5.) Moreover, it will be proved in Section 5 that an indefinite integral F is an a-primitive of f ; that is, $F'(x) = f(x)$ almost everywhere on the interval.

For the moment we will focus our attention on the differentiation of an indefinite integral at a specific point $c \in [a,b]$. We will see that (one-sided) continuity of f at a point c implies (one-sided) differentiability of any indefinite integral at c. We recall that saying f has a right hand limit A at $c \in [a,b)$ means that given $\varepsilon > 0$, there exists $\eta > 0$ with $\eta \le b - c$ such that if x belongs to $(c, c+\eta)$, then

$$(4.\zeta) \qquad A - \varepsilon \le f(x) \le A + \varepsilon.$$

We leave it to the reader to formulate the definition of a left hand limit of a function, and to state and prove a "left hand version" of the following result.

4.6 FUNDAMENTAL THEOREM II.

Let $f \in R^*([a,b])$ and let f have a right hand limit A at a point $c \in [a,b)$. Then the indefinite integral

$$F_u(x) := \int_u^x f$$

has a right hand derivative at c equal to A.

Proof. We will consider the case $u = a$ and denote F_a by F, leaving the general case as an exercise for the reader.

Let h satisfy $0 < h < \eta$. Since f is integrable on the intervals $[a,c],[a,c+h]$, and $[c,c+h]$ (by Corollary 3.8), we have $F(c+h) - F(c) = \int_c^{c+h} f$. Now on the interval $(c,c+h]$ the function f satisfies $(4.\zeta)$, so that (see Corollary 3.4 and Exercise 3.D) we have $(A-\varepsilon)h \le \int_c^{c+h} f \le (A+\varepsilon)h.$. It follows that

$$\left| \frac{F(c+h) - F(c)}{h} - A \right| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{h \to 0+} \frac{F(c+h) - F(c)}{h} = A;$$

this means that F has a right hand derivative $F_+'(c)$ at c which is equal to A.

Corollary. Let $f \in R^*([a,b])$ be continuous at $c \in [a,b]$. Then the indefinite integral F_u of f is differentiable at c and $F_u^{'}(c) = f(c)$.

Proof. Let $c \in (a,b)$. If f is continuous at c, then both the left and right limits of *f* equal $f(c)$. Consequently, both the left and right derivatives of F_u. at c exist and equal $f(c)$. Similarly for the endpoints. We recall from Definition 4.1(c) that a function F is said to be an in- definite integral of $f \in R^*(I)$ in case $F - F_a$ is a constant function. The preceding corollary implies that if *f* is continuous at a point $c \in I$, then $F^{'}(c) = F_a^{'}(c) = f(c)$. We can reformulate that corollary in the following statement.

Corollary. Let f be continuous on $I := [a,b]$. Then any indefinite integral F of f is differentiable on I and $F^{'}(x) = f(x)$ for all $x \in I$.

Proof. Apply the preceding corollary to each point of I.

We can restate Corollary 4.10 in the form: If *f* is continuous on $[a,b]$, then any indefinite integral of *f* is a primitive of *f* on $[a,b]$. We now state a much deeper theorem about the differentiation of the integral; its proof is delicate and will be given in Section 5, where we will give a complete characterization of indefinite integrals of functions in $R^*([a,b])$.

Check Your Progress

1. Prove: Let $F : I \to R$ be *differentiable at* a point $t \in I$. *Given* $\varepsilon > 0$ there *exists* $\delta_\varepsilon(t) > 0$ *such that if* $u, v \in I$ *satisfy*

$$t - \delta_\varepsilon(t) \le u \le t \le v \le t + \delta_\varepsilon(t),$$

then we have

$$|F(v) - F(u) - F^{'}(t)(v-u)| \le \varepsilon(v-u). \qquad (4.\beta)$$

2. Prove: If $f:[a,b]\to R$ has a primitive F on $[a,b]$, then $f\in R^*([a,b])$ and $(4.\gamma)$ $\int_a^b f = F(b)-F(a).$

3. Prove: If $f:[a,b]\to R$ has a c-primitive F on $[a,b]$, then $f\in R^*([a,b])$ and $(4.\varepsilon)$ $\int_a^b f = F(b)-F(a).$

4. Prove: Let $f\in R^*([a,b])$ and let f have a right hand limit A at a point $c\in[a,b)$. Then the indefinite integral $F_u(x):=\int_u^x f$ has a right hand derivative at c equal to A.

4.7 LET US SUM UP

. Let $I:=[a,b]\subset R$ and let $F,f:I\to R.$

(a) We say that F is a primitive (or an antiderivative) of *f* *on I* if the derivative $F'(x)$ exists and $F'(x)=f(x)$ *for all* $x\in I.$

2. Let $F:I\to$ be *differentiable at* a point $t\in I.$ *given* $\varepsilon>0$ there *exists* $\delta_\varepsilon(t)>0$ *such that if* $u,v\in I$ *satisfy*

$$t-\delta_\varepsilon(t)\le u\le t\le v\le t+\delta_\varepsilon(t),$$

then we have

$$(4.\beta)\qquad \left|F(v)-F(u)-F'(t)(v-u)\right|\le\varepsilon(v-u).$$

3. If $f:[a,b]\to R$ has a primitive F on $[a,b]$, then $f\in R^*([a,b])$ and

$$(4.\gamma)\qquad \int_a^b f=F(b)-F(a).$$

4. If $f:[a,b]\to R$ has a c-primitive F on $[a,b]$, then $f\in R^*([a,b])$ and

$$(4.\varepsilon)\qquad \int_a^b f=F(b)-F(a).$$

5. Let $f\in R^*([a,b])$ and let f have a right hand limit A at a point $c\in[a,b)$. Then the indefinite integral

$$F_u(x):=\int_u^x f$$

has a right hand derivative at c equal to A.

6. If sgn is the signum function on R defined by

$$\operatorname{sgn}(x):=\begin{cases}-1 & if\quad x<0,\\ 0 & if\quad x=0,\\ +1 & if\quad x>0,\end{cases}$$

then sgn is integrable over any interval $[a,b]\subset R,$ since its restriction to this interval is a step function. It is easy to see that the function $H(x):=|x|$ *is* an f-primitive of sgn on any interval $[a,b]$ with exceptional set $E=\{0\}$

4.8 KEY WORDS

Straddle Lemma

Step function

Integrative derivation

Sign function

Fundamental theorem

4.9 QUESTIONS FOR REVIEW

1. Prove fundamental theorem-1

2.Prove fundamental theorem-2

4.10 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

4.11 ANSWERS TO CHECK YOUR PROGRESS

1. See section 4.3

2. See section 4.4

3. See section 4.5

UNI-5 THE FUNDAMENTAL THEOREMS OF CALCULUS -2

STRUCTURE

5.0 Objective

5.1 Introduction

5.2 Fundamental theorem-III

5.3 The Cantor set

5.4 A characterization of Indefinite Integrals

5.5 Let us sum up

5.6 Key words

5.7 Questions for review

5.8 Suggestive readings and references

5.9 Answers to check your progress

5.0 OBJECTIVE

In this unit we will learn and understand about Fundamental theorem of calculus –III, The Cantor set, A characterization of Indefinite Integrals

5.1 INTRODUCTION

The fundamental theorem of calculus is a theorem that links the concept of differentiating a function with the concept of integrating a function.

The first part of the theorem, sometimes called the first fundamental theorem of calculus, states that one of the antiderivatives (also called *indefinite integral*), say *F*, of some function *f* may be obtained as the integral of *f* with a variable bound of integration. This implies the existence of antiderivatives for continuous functions.[1]

Notes

Conversely, the second part of the theorem, sometimes called the second fundamental theorem of calculus, states that the integral of a function *f* over some interval can be computed by using any one, say *F*, of its infinitely many antiderivatives. This part of the theorem has key practical applications, because explicitly finding the antiderivative of a function by symbolic integration avoids numerical integration to compute integrals. This provides generally a better numerical accuracy.

5.2 Fundamental Theorem III.

If $f \in R^*(I)$ where $I := [a,b]$, then any indefinite integral F is continuous on I and is an a-primitive of *f* on $[a,b]$. Thus, there exists a null set $Z \subset I$ such that

$$(4.\eta) \qquad F'(x) = f(x) \quad \text{for all} \quad x \in I - Z.$$

Unfortunately, the-preceding theorem does not assert that an indefinite integral of a function $f \in R^*([a,b])$ is a c-primitive of *f*; see Example 4.18(c). The next result is a useful one; it establishes the existence of a c-primitive for a. large and important class of functions.

Theorem. If $f : [a,b] \to R$ is a regulated function, then any indefinite integral of *f* is a c-primitive of *f* on $[a,b]$.

Proof. We saw in Theorem 3.20 that if f is a regulated function, then there exists a countable set D such that *f* is continuous at every point of I - D. It follows from 4.11 that the indefinite integral $F_u(x) = \int_u^x f$ is continuous on I, and from Corollary 4.9 that it is differentiable at every point $c \in I - D$ and that $F_u'(c) = f(c)$. Therefore, F_u is a c-primitive of *f* on I.

Some Remarks

We now offer two sets of remarks that are intended to clarify the rather subtle distinction between c-primitives and indefinite integrals.

Remarks. (a) An R*-integrable function always has indefinite integrals, and every indefinite integral of a function in R*(I) is an a-primitive.

(b) An R*-integrable function does not always have a c-primitive; see Example. However, every continuous function has a primitive and every regulated function has a c-primitive.

(c) If F is a c-primitive of $f: I \to R$, then $f \in R^*(I)$ and F is an indefinite integral of *f*.

(d) If F is an a-primitive of $f \in R^*(I)$, then F need not be an indefinite integral of f .

Remarks. In discussing c-primitives, the exceptional set (where $F'(x) = f(x)$ does not hold) is a countable set, while in Theorem 4.11 the exceptional set *Z* is a null *set.* Now every countable set is a null set, but the converse is not true. as we will see in Theorem 4.16. It is natural to ask whether this gap between a countable set and a null set of exceptional points can be bridged. There are two parts to this question:

(a) Can we replace Theorem by the assertion: If *F* is a continuous function on $I := [a,b]$ and there exists a null set *Z* such that $F'(x) = f(x)$ for all $x \in I - Z$, then $f \in R^*([a,b])$ and $\int_a^b f = F(b) - F(a)$? [That is, if *F* is an a-primitive of $f:[a,b] \to R$,, then is $f \in R^*([a,b])$ and $\int_a^b f = F(b) - F(a)$? *]*

(b) Can we replace Theorem , by the assertion: If *F* is an indefinite integral of $f \in R^*([a,b])$, then there exists a countable set *C* such that $F'(x) = f(x)$ for all $x \in I - C$? [In other words, if *F is* an indefinite integral of *f* , then is *F* a c-primitive of *f* ?*]*

The answer to both of these questions is: No. However, in order to establish this claim, we will construct the Cantor set and the Cantor-Lebesgue singular function, both of which will be useful later.

5.3 THE CANTOR SET

We will construct a subset of $I := [0,1]$ by a process of successively removing open middle thirds. We start with *I* and obtain the set Γ_1 by removing the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ to obtain

$$\Gamma_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Next we remove the open middle thirds of the two intervals in Γ_1 to obtain

$$\Gamma_2 := \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

The set Γ_3 is obtained by removing the middle thirds of each of the 2^2 *sets in* Γ_2; thus Γ_3 consists of 2^3 closed intervals, each having length $1/3^3$. Continuing *in* this way, we obtain Γ_n as the union of 2^n intervals of the form $\left[k/3^n, (k+1)/3^n\right]$. Note the first few stages of this construction, as indicated in Figure 5.1.

Definition. The Cantor set Γ is the intersection of the decreasing sequence of sets $\Gamma_n, n \in R$, obtained in this way.

Historical note. Recently, K. Hannabuss *[Math. Intelligencer* 18 (1996), no. 3, 28-31] has pointed out that what is universally called the Cantor set appeared in an 1875 paper by H. J. S. Smith, some eight years before Cantor mentioned it. See also Hawkins [Hw-1; p. 38].

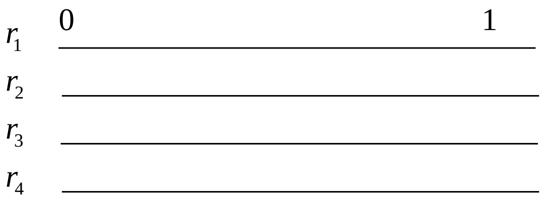

Figure 5.1 Construction of the Cantor set.

Theorem. The Cantor set Γ is an uncountable null set.

Proof. In fact, the set Γ_n is the union of 2^n closed intervals, each of which has length $1/3^n$. If $\varepsilon > 0$ is given, let n_0, be such that $(2/3)^{n_0} < \varepsilon$. Since $\Gamma \subset \Gamma_{n_0}$, then Γ is contained in the union of a finite number of closed intervals with total length $< \varepsilon$. It follows from Exercise 2.M that Γ is a null set.

Assume that Γ is a countable set and let $\{x_n : n \in \mathbb{N}\}$ be anenumeration of it. Let I_1 be the closed interval of length 1/3 in Γ_1 such that $x_1 \notin I_1$. If $n \geq 2$, let I_n be the first interval in $I_{n-1} \cap \Gamma_n$ having length $1/3^n$ such that $x_1 \notin I_1$. In this way, we obtain a nested sequence (I_n) of compact intervals; invoking the Nested Intervals Theorem [B-S; p . 46], we obtain a point $z \in \bigcap_{k=1}^{\infty} I_k$ such that $z \in \Gamma$. Since $x_k \notin I_k$, we conclude that $z \neq x_k$ for all $k \in \mathbb{N}$. Therefore the above enumeration does not exhaust Γ, and this set is not countable. We will now construct a function $\Lambda : [0,1] \to R$ that is often useful in constructing examples and counterexamples. First let Λ_1 be the piecewise linear function with $\Lambda_1(0) := 0, \Lambda_1(x) := \frac{1}{2}$ for $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $\Lambda_1(1) := 1$. Next, let Λ_2 be the piecewise linear function with $\Lambda_2(0) := 0$, taking the values 1/4,1/2, and 3/4 on the intervals

$$\left[\frac{1}{9}, \frac{2}{9}\right], \qquad \left[\frac{1}{3}, \frac{2}{3}\right], \qquad \left[\frac{7}{9}, \frac{8}{9}\right],$$

respectively, and $\Lambda_2(1) := 1$. In general, Λ_n is the piecewise linear function with $\Lambda_n(0) := 0$, taking the values $1/2^n, 2/2^n, \ldots\ldots, (2^n - 1)/2^n$ on the closed intervals corresponding to the intervals that were removed to construct Γ_n, and with $\Lambda_n(1) := 1$. By definition, each Λ_n, is an increasing (i.e., = non- decreasing) continuous function. We claim that this sequence of functions converges on $[0,1]$ to a limit function, which we call the Cantor-Lebesgue singular function and denote by Λ. (Sometimes the graph of Λ is called "the Devil's staircase" .)

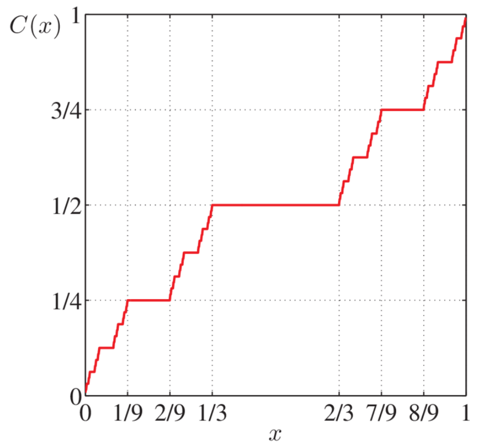

Figure 5.2 Construction of the Cantor-Lebesgue singular function.

Theorem. The Cantor-Lebesgue singular function $\Lambda : [0,1] \to R$ is continuous and increasing on $[0,1]$ and its derivative $\Lambda'(x) = 0$ for all points $x \in [0,1] - \Gamma$.

Proof. Since the graphs of Λ_n and Λ_{n+1} either coincide or lie in the same horizontal strips with thickness $1/2^n$, we have $|\Lambda_n(x) - \Lambda_{n+1}(x)| \le 1/2^n$ for all $n \in R, x \in [0,1]$. Therefore, if $m > n$ it then

$$|\Lambda_n(x) - \Lambda_m(x)| \le \sum_{k=n}^{m-1} |\Lambda_k(x) - \Lambda_{k+1}(x)| \le \sum_{k=n}^{m-1} \frac{1}{2^k} \le \frac{1}{2^{n-1}}.$$

This implies that the limit

$$\Lambda(x) := \lim_{n\to\infty} \Lambda_n(x)$$

exists and that $|\Lambda_n(x) - \Lambda(x)| \le 1/2^{n-1}$ for all $x \in [0,1]$. Therefore the sequence (Λ_n) converges uniformly on $[0,1]$ to Λ. Since each Λ_n is continuous, it follows (see [B-S; p. 234]) that A is continuous on $[0,1]$. Since each Λ_n is increasing, we also conclude that Λ is increasing on $[0,1]$.

If $X \in [0,1] - \Gamma$, then there is an open interval containing x on which all of the functions Λ_n are constant (and equal) for sufficiently large *n*. Therefore Λ is constant on this open interval and $\Lambda^{'}(x) = 0$.

Examples. (a) We return to the question raised in -1.14(a). We have seen that the Cantor-Lebesgue singular function Λ is a continuous function on $[0,1]$ and that $\Lambda'(X) = 0$ for all $x \in [0,1] - \Gamma$. Since Γ is a null set, then A is an a-primitive of the 0-function on [0. 1]. However, $\int_0^1 \Lambda' = 0 \ne 1 = \Lambda(1) - \Lambda(0)$, showing that the answer to 5.1.1(a) is: No.

(b) We return to the question raised in 4.14(b). We $\varphi(x) := 1$ for $x \in \Gamma$ and $\varphi(x) := 0$ for $x \in [0,1] - \Gamma$. Since the Cantor set Γ is a null set, the function φ is a null function. and Example 2.6(a) implies that $\varphi \in R^*([0,x])$ and $\Phi(x) := \int_0^x \varphi = 0$

for all $x \in [0,1]$. Consequently, $\Phi'(x) = 0$ for all $x \in [0,1]$. However, $\varphi(x) = 1$ for $x \in \Gamma$, so that $\Phi'(x) \ne \varphi(x)$ on an uncountable null set.

(c) The example in (b) shows that, while an integrable function does have an a-primitive, it does not always have a c-primitive.

Indeed, if Ψ were a c-primitive of φ, then $\Psi - \Psi(0) = \Phi$, so $\Psi'(x) = 0$ for all $x \in [0,1]$. But the hypothesis that Ψ is a c-primitive of φ implies that $\varphi(x) \neq 0$ only on a countable set, contrary to the fact that the Cantor set Γ is not countable. As was seen in Theorem 4.16.

5.4 A CHARACTERIZATION OF INDEFINITE INTEGRALS

The remarks just made show that there is some delicacy in the identification of indefinite integrals of functions in $R^*(I)$.

However, in Section 5 we will give a complete characterization for indefinite integrals of (generalized Riemann) integrable functions. We will see that a function F is the indefinite integral of a function in $R^*(I)$ if and only if F s differentiable a.e., and that on the null set Z where F is not differentiable, the function F satisfies an additional condition which is automatically satisfied if Z is a countable set.

Integration by Parts

We now give a weak (but often useful) form of the Integration by Parts formula. Stronger forms of this result will be given in Section 12.

Theorem. Let F and G be differentiable on $I := [a,b]$. Then FIG belongs to $R^*(I)$ if and only if FG' belongs to $R^*(I)$. In this case, we have

$$(4.\theta) \qquad \int_a^b F'G = FG\Big|_a^b - \int_a^b FG'.$$

Proof. The Product Theorem from calculus asserts that $(FG)' \in R^*(I)$, and it follows from equation $(4.l)$ that $F'G \in R^*(I)$ if and only if $FG' \in R^*(I)$. Formula $(4.\theta)$ now follows immediately.

Some Examples

We will conclude this section by giving three examples of functions that have a c-primitive, but not an f-primitive.

Examples. (a) Let f be the Dirichlet function defined on $[0,1]$ by $f(x) := 0$ if x is irrational, and $f(x) := 1$ if $x \in \mathbb{Q} \cap [0,1]$.

It was seen in Example 2.3 (a) that $f \in R^*([0,1])$. It is an exercise to show that the zero function $F(x) := 0$ for all $x \in [0,1]$ is a c-primitive of f.

(b) As in Example 2.7, Let $c_k := 1 - 1/2^k$ for $k = 0,1,\ldots,$ so that $c_0 = 0, c_1 = \frac{1}{2}, c_2 = \frac{3}{4}, \ldots\ldots$ We let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1 & if \quad x \in [c_{2k}, c_{2k+1}), k = 0,1,\ldots, \\ 0 & if \quad x \in [c_{2k+1}, c_{2k+2}), k = 0,1,\ldots, \\ 0 & if \quad x = 1. \end{cases}$$

See Figure 4.3 for a graph of f.

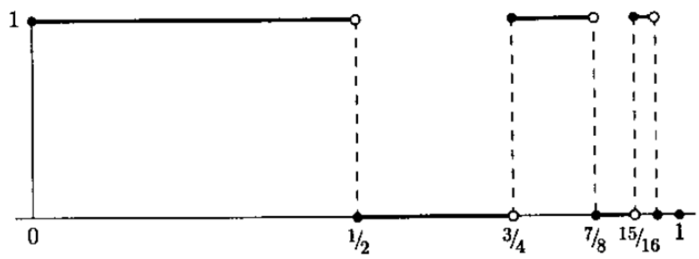

Figure 5.3 Graph of *f* .

We will show that $f \in R^*([0,1])$ by exhibiting a c-primitive of f. In fact, we define $F : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & if \quad x = 0, \\ x - c_{2k} + F(c_{2k}) & if \quad x \in (c_{2k}, c_{2k+1}], k = 0,1,\ldots, \\ F(c_{2k+1}) & if \quad x \in (c_{2k+1}, c_{2k+2}], k = 0,1,\ldots, \\ \frac{2}{3} & if \quad x = 1. \end{cases}$$

Thus we have $F(x)=x$ for $x\in\left[0,\frac{1}{2}\right], F(x)=\frac{1}{2}$ for $x\in\left[\frac{1}{2},\frac{3}{4}\right], F(x)=x-\frac{3}{4}+\frac{1}{2}=x-\frac{1}{4}$ for $x\in\left[\frac{3}{4},\frac{7}{8}\right], F(x)=\frac{1}{8}+\frac{1}{2}=\frac{5}{8}$ for $x\in\left[\frac{7}{8},\frac{15}{16}\right]$, etc. See Figure 4.4 for a graph of F.

An elementary induction argument shows that

$$F(c_{2k+1})=F(c_{2k})+\frac{1}{2^{2k+1}}=F(c_{2k-1})+\frac{1}{2^{2k+1}},$$

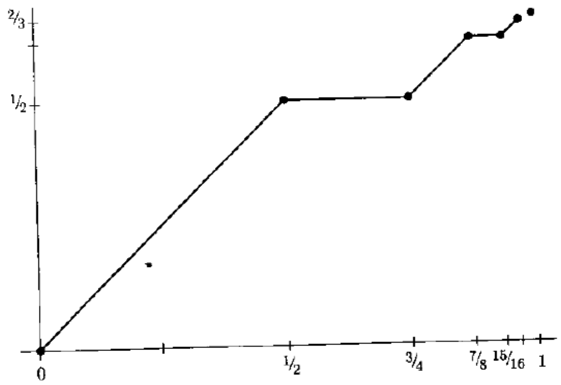

Figure 5.4. Graph of F.

so that we have

$$F(c_{2k+1})=\frac{1}{2}+\frac{1}{2^3}+.....+\frac{1}{2^{2k+1}}=\frac{2}{3}\left(1-\frac{1}{4^k}\right).$$

We claim that F is continuous on $[0,1)$. Indeed, this is obvious at every point $x\neq c_k, k\in R$. Further, by considering the cases c_{2k} and c_{2k+1} separately, one determines that both the left and right hand limits exist at these points and equal $F(c_{2k})$ and $F(c_{2k+1})$, respectively. To see that ***F*** is continuous from the left at $x=1$, one can show that ***F*** is increasing on $[0,1]$ and that lim $F(c_{2k+1})=2/3=F(1)$.

From the definition of *F* it is evident that the derivative $F'(x)=1$ when $x\in(c_{2k},c_{2k+1})$, and that $F'(x)=0$ when

$x \in (c_{2k+1}, c_{2k+2})$. It is also clear that F does not have a two-sided derivative at any of the points $c_k, k = 0,1,\ldots\ldots$ Therefore, *F is* a c-primitive of *f*, but it is not an f-primitive.

It follows from the Fundamental Theorem 5.3 that $f \in R^*([0,1])$ and that

$$\int_0^1 f = F(1) - F(0) = \frac{2}{3}.$$

(c) We now consider the function $\Psi(x) := x|\cos(\pi/x)|$ for $x \in (0,1]$ and $\Psi(0) := 0$. It is clear that Ψ is continuous on $[0.1]$. Moreover, $\Psi(a) = 0$ if and only if $a \in E := \{0\} \cup \{2/(2k+1) : k \in \mathbb{N}\}$. For a graph of Ψ Figure 5.5 .

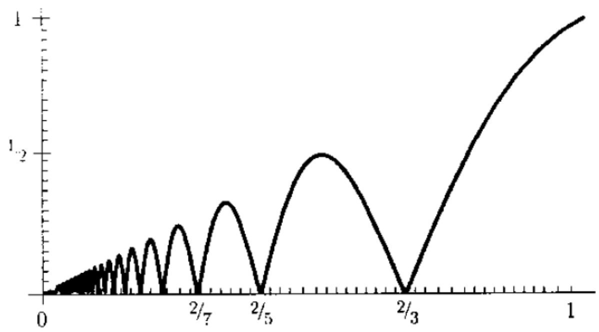

Figure 5.5 Graph of Ψ.

Direct calculation shows that the derivative $\Psi'(0)$ does not exist, since $\Psi(1/k) = 1/k$, while $\Psi(2/(2k+1)) = 0$. To investigate the existence of $\Psi'(x)$ elsewhere on $(0,1]$, we recall that the absolute. value function $x \mapsto |x|$ has a derivative equal to *sgn x when* $x \neq 0$. Therefore the Chain Rule implies that the function $x \mapsto |\cos(\pi/x)|$ has a derivative when *x* is not a zero of $\cos(\pi/x)$; that is, when $x \neq 2/(2k+1)$. The Product Rule for differentiation then shows that Ψ has a derivative for $x \notin E.$. Moreover, it can be

Notes

shown that Ψ does not have a derivative when $x \in E$. (For a related discussion, see [B-S; p. 163].)

Thus, if we let $\psi(x) := \Psi'(x)$ for $x \notin E$ and $\psi(x) := 0$ for $x \in E$, then Ψ is a c-primitive of ψ, whence it follows that $\psi \in R^*([0,1])$.

Check Your Progress

1. Prove: If $f:[a,b] \to R$ is a regulated function, then any indefinite integral of *f* is a c-primitive of *f* on $[a,b]$.

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2. Explain about the Cantor set.

The Cantor set Γ is an uncountable null set.

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5.5 LET US SUM UP

1. If $f \in R^*(I)$ where $I := [a,b]$, then any indefinite integral F is continuous on I and is an a-primitive of *f* on $[a,b]$. Thus, there exists a null set $Z \subset I$ such that

$$(4.\eta) \qquad F'(x) = f(x) \text{ for all } x \in I - Z.$$

2. If $f:[a,b] \to R$ is a regulated function, then any indefinite integral of *f* is a c-primitive of *f* on $[a,b]$.
3. The Cantor set Γ is an uncountable null set.
4. The Cantor-Lebesgue singular function $\Lambda:[0,1] \to R$ is continuous and increasing on $[0,1]$ and its derivative $\Lambda'(x) = 0$ for all points $x \in [0,1] - \Gamma$.

5.6 KEY WORDS

The Cantor set

A characterization of indefinite integrals

Cantor-Lebesgue singular function

5.7 QUESTIONS FOR REVIEW

1. Prove fundamental theorem-III
2. Explain about Cantor set
3. Explain about a characterization of Indefinite Integrals

5.8 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle
2. The elements of Integration and Lebesgue Meassure
3. A course on integration- Nicolas Lerner
4. General theory of Integration- Dr. E.W. Hobson
5. General theory of Integration- P.Muldowney
6. General theory of functions and Integration- Angus E.Taylor

5.9 ANSWERS TO CHECK YOUR PROGRESS

1. See section 5.3
2. See section 5.4
3. See section 5.4

UNIT-6 THE SAKS-HENSTOCK LEMMA

STRUCTURE

6.0 Objective

6.1 Introduction

6.2 The Saks-Henstock Lemma

6.3 Saks-Henstock lemma

6.4 Continuity of the Indefinite integrals

6.5 Characterization of null functions

6.6 Let us sum up

6.7 Key words

6.8 Questions for review

6.9 Suggestive readings and references

6.10 Answers to check your progress

6.0 OBJECTIVE

In this unit we will learn and understand about The Sakes- Henstock lemma, Continuity of the indefinite integrals, Characterization of null functions

6.1 INTRODUCTION

The theory of integration has its ancient and honorable roots in the "method of exhaustion" that was invented by Eudoxos and greatly developed by Archimedes for the purpose of calculating the areas and volumes of geometric figures. The later work of Newton and Leibniz enabled this method to grow into a systematic tool for such calculations.

As this theory developed, it become less concerned with applications to geometry and elementary mechanics, for which it is entirely adequate, and more concerned with purely analytic questions, for which the classical theory of integration is not always sufficient. Thus a present-day mathematician is apt to be interested in the convergence of orthogonal expansions, or in applications to differential equations or probability. For him the classical theory of integration which culminated in the Riemann integral has been largely replaced by the theory which has grown from the pioneering work of Henri Lebesgue at the beginning of this century. The reason for this is very simple: the powerful convergence theorems associated with the Lebesgue theory of integration lead to more general, more complete, and more elegant results than the Riemann integral admits.

Lebesque's definition of the integral enlarges the collection of functions for which the integral is defined. Although this enlargement is useful in itself, its main virtue is that the theorems relating to the interchange of the limit and the integral are valid under less stringent assumptions than are required for the Riemann integral. Since one frequently needs to make such interchanges, the Lebesque integral is more convenient to deal with than the Riemann integral. To exemplify these remarks, let the sequence (f_n) of functions be defined for $x > 0$ by $f_n(x) = e^{-nx} / \sqrt{x}$. It is readily seen that the (improper) Riemann integrals

$$I_n = \int_0^{+\infty} \frac{e^{-nx}}{\sqrt{x}} dx$$

Exist and that $\lim_{n\to\infty} f_n(x) = 0$ for all $x > 0$. however, since $\lim_{x\to 0} f_n(x) = +\infty$ for each n, the convergence of the sequence is certainly not uniform for $x > 0$. Although it is hoped that the reader can supply the estimates required to show that $I_n = 0$, we prefer to obtain this conclusion as an immediate consequence of the Lebesgue Dominated Convergence Theorem which will be proves later. As another example, consider the function F defined for $t > 0$ by the (improper) Riemann integral

$$F(t) = \int_0^{+\infty} x^2 e^{-tx} dx.$$

With a little effort one can show that F is continuous and that its derivative exists and is given by

$$F(t) = -\int_0^{+\infty} x^3 e^{-tx} dx,$$

Which is obtained by differerntiating under the integral sign. Once again, this inference follows easily form the Lebesgue Dominated Convergence Theorem.

At the risk of oversimplification, we shall try to indicate the crucial difference between the Riemann and the Lebesgue definitions of the integral. Recall that an interval in the set R of real numbers is a set which has one of the following four forms:

$$[a,b] = \{x \in R :\leq x \leq b\}, \quad (a,b) = \{x \in R : a < x < b\},$$

$$[a,b] = \{x \in R :\leq x < b\}, \quad (a,b) = \{x \in R : a < x \leq b\}.$$

In each of these cases we refer to a and b as the endpoints and prescribe b-a as the length of the interval. Recall further that if E is a set, then the characteristic function of E is the function X_E defined by

$$\begin{aligned} X_E(x) &= 1, \quad \text{if } x \in E, \\ &= 0, \quad \text{if } x \notin E. \end{aligned}$$

A step function is a function φ which is a finite linear combination of characteristic functions of intervals; thus

$$\varphi = \sum_{j=1}^{n} c_j X_{Ej}.$$

If the endpoints of the interval E_j are a_j, b_j, we define the integral of φ to be

$$\int \varphi = \sum_{j=1}^{n} c_j \left(b_j - a_j\right).$$

If f is a bounded function defined on an interval $[a,b]$ and if f is not too discontinuous, then the Riemann integral of f is defined to be the limit (in an appropriate sense) of the integrals of step functions which approximate f. In particular, the lower Riemann integral of f may be defined to be the supremum of the integrals of all step functions φ such that $\varphi(x) \le f(x)$ for all x in $[a,b]$, and $\varphi(x) = 0$ for x not in $[a,b]$.

The Lebesgue integral can be obtained by a similar process, expect that the collection of step functions is replaced by a larger class of functions. In somewhat more detail, the notion of length is generalized to a suitable collection X of subsets of R. Once this is done, the step functions are replaced by simple functions, which are funite linear combinations of characteristic functions of sets belonging to X. If

$$\varphi = \sum_{j=1}^{n} c_j X_{Ej}$$

Is such a simple function and if $\mu(E)$ denotes the “measure” or “generalized length” of the set E in X, we define the integral of $\varphi(E)$ to be

$$\int \varphi = \sum_{j=1}^{n} c_j \mu\left(E_j\right).$$

If f is a nonnegative function defined on R which is suitably restricted, we shall define the (Lebesgue) integral of f to be the supremum of the integrals of all simple functions $\varphi(x) \le f(x)$ for all x in R. The integral can then be extended to certain functions that take both signs.

Although the generalization of the notion of length to certain sets in R which are not necessarily intervals has great interest, it was observed in 1915 by Maurice Frechet that the convergence properties of the Lebesgue integral are valid in considerable generality. Indeed, let X be any set in which there is a collection X of subsets containing the empty set ϕ and X and closed under complementation and countable unions. Suppose that there is a nonnegative measure function μ defined on X such that $\mu(\phi) = 0$ and which is countably additive in the sense that

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

For each sequence (E_j) of sets in X which are mutually disjoint. In this case an integral can be defined for a suitable class of real-valued functions on X, and this integral possesses strong convergence properties.

As we have stressed, we are particularly interested in these convergence theorems. Therefore we wish to advance directly toward them in this abstract setting, since it is more general and, we believe, conceptually simplier than the special cases of integration on the line or in R^n. However, it does require that the reader temporarily accept the fact that interesting special cases are subsumed by the general theory. Specifically, it requires that he accept the assertion that there exists a countably additive measure function that extends the notion of the length of an interval. The proof of this assertion if in Chapter 9 and can be read after completing Chapter 3 by those for whom the suspense is too great.

In this introductory chapter we have attempted to provide motivation and to set the stage for the detailed discussion which follows. Some of our remarks here have been a bit vague and none of them has been proved. These defects will be remedied. However, since we shall have occasion to refer to the system of extended real numbers, we now append a brief description of this system.

In integration theory it is frequently convenient to adjoin the two symbols $-\infty, +\infty$ to the real number system R. (It is stressed that these symbols are not real numbers.) We also introduce the convention that $-\infty < x < +\infty$ for any $x \in R$. The collection $\bar{R}$ consisting of the set $R \cup \{-\infty, +\infty\}$ is called the extended real number system.

One reason we wish to consider $\bar{R}$ is that it is convenient to say that the length of the real line is equal to $+\infty$. Another reason is that we will frequently be talking the supremum (= least upper bound) of a set of real numbers. We know that a nonempty set A of real numbers which has an upper bound also has a supremum (in R). If we define the supremum of a nonempty set which does not have an upper bound to be $+\infty$, then every nonempty subset of R $\left(\text{or}\,\bar{R}\right)$ has a unique supremum in $\bar{R}$. Similarly, every nonempty subset of R $\left(\text{or}\,\bar{R}\right)$ has a unique infimum (= greatest lower bound) in $\bar{R}$. (Some authors introduce the conventions that $\inf\phi = +\infty, \sup\phi = -\infty,$ but we shall not employ them.)

If $\left(x_n\right)$ is a sequence of extended real numbers, we define the limit superior and the limit inferior of this sequence by

$$\lim\sup x_n = \inf_m\left(\sup_{n\geq m} x_n\right),$$

$$\lim\inf x_n = \sup_m\left(\inf_{n\geq m} x_n\right).$$

If the limit inferior and the limit superior are equal, then their value is called the limit of the sequence. It is clear that this agress with the conventional definition when the sequence and the limit belong to R.

Finally, we introduce the following algebraic operations between the symbols $\pm\infty$ and elements $x \in R$:

$$\begin{aligned}
(\pm\infty)+(\pm\infty) = x+(\pm\infty) = (\pm\infty)+x &= \pm\infty,\\
(\pm\infty)(\pm\infty) = +\infty, (\pm\infty)(\mp\infty) &= -\infty,\\
x(\pm\infty) = (\pm\infty)x = \pm\infty &\quad \text{if } x > 0,\\
= 0 &\quad \text{if } x = 0,\\
= \mp\infty &\quad \text{if } x < 0.
\end{aligned}$$

It should be notices that we do not define $(+\infty)+(-\infty)$ or $(-\infty)+(+\infty)$, nor do we define quotients when the denominator is $\pm\infty$.

6.2 THE SAKS- HENSTOCK LEMMA

Notes

The Saks-Henstock Lemma, sometimes called the "Henstock Lemma", is of fundamental importance in proving the deeper properties of the (generalized Riemann) integral. Henstock [H-5; p.197] attributes this result to Saks [S-1; p.214], but its use for the general integral is certainly due to Henstock. Our first use of this lemma is to prove the continuity of the indefinite integral. We will then establish the eifferentiabnility almost everywhere of the indefinite integral, announced in Theorem 4.11; however, in order to prove this result we also require the important Vitali Covering Theorem. Next we give a characterization of null functions. Finally, we present a necessary and sufficient condition for a function to be an indefinite integral of a function in $R^*([a,b])$.

This section contains some rather subtle arguments. The reader may want to look over the results and defer a detailed reading until a later time.

The definition of the integral of a function *f* on $I := [a,b]$ requires that, given $\varepsilon > 0$ there exists a gauge δ_ε on I such that if $\dot{P}$ is any δ_ε-fine partition of I, then the Riemann sum $S(f;\dot{P})$ satisfies the inequality $\left|S(f;\dot{P}) - \int_a^b f\right| \le \varepsilon$. The Saks-Henstock Lemma asserts that the same degree of approximation is valid for the difference between any subset of terms from this Riemann sum and the sum of the integrals of f over the corresponding subintervals. This fact may not seem so surprising when the subintervals in the subset of $\dot{P}$ consist of abutting intervals. However, it is not at all obvious that the result remain true for an arbitrary collection of subintervals. Even more surprising is that we can replace the absolute value of the sum of these differences by the sum of the absolute values and still have essentially the same degree of approximation. This, despite the fact that the existence of the integral may depend on the subtraction of terms in the Riemann sums.

6.1. Definition. Let $I := [a,b]$ be a nondegenerate compact interval.

(a) A subpartition of I is a collection $\{J_j\}_{j=1}^{s}$ of nonoverlapping closed intervals in I.

(b) A tagged subpartition of I is a collection $\dot{P}_0 = \{(J_j, t_j)\}_{j=1}^{s}$ of ordered pairs, consisting of intervals $\dot{P}_0 = \{J_j\}_{j=1}^{s}$ that form a subpartition of I, and tags $t_j \in J_j$ for $j = 1, \ldots\ldots, s.$

(c) If δ is a gauge on I, we say that the tagged subpartition $\dot{P}_0$ is δ-fine if $J_j \subseteq [t_j - \delta(t_j), t_j + \delta(t_j)]$ for $j = 1, \ldots, s.$

(d) If δ is a gauge on a subset $E \subseteq I,$ we say that the tagged subpartition $\dot{P}_0$ is (δ, E)-fine if all tags $t_j \in E$ and $J_j \subseteq [t_j - \delta(t_j), t_j + \delta(t_j)]$ for $j = 1, \ldots, s.$

6.2 Remarks. (a) Any subset of a partition of I is a subpartition of I. Conversely, it is an exercise to show that if Π_0 is a subpartition of I, then there exists a partition of I of which Π_0 is a subset.

(b) If $\dot{P}_0$ is a subpartition of I that is δ-fine, then it is an exercise to show that there exists a δ-fine partition of I of which $\dot{P}_0$ is a subset.

(c) Definition 6.1 (d) only requires that δ be defined on E, but one can set $\delta(x) := 1$ for $x \in I - E$ and obtain a gauge on all of I.

If $\dot{P}_0 = \{(J_j, t_j) : j = 1, \ldots, s\}$ is a tagged subpartition of I, then we let $U(\dot{P}_0) := \bigcup_{j=1}^{s} J_j.$ If $f \in R^*(I)$, we define

$$S(f; \dot{P}_0) := \sum_{j=1}^{s} f(t_j) l(J_j) \qquad \text{and} \qquad \int_{U(\dot{P}_0)} f := \sum_{j=1}^{s} \int_{J_j} f,$$

Where $l(J)$ denotes the length of the interval J.

6.3 SAKS-HENSTOCK LEMMA.

Notes

Let $f \in R^*([a,b])$ and for $\varepsilon > 0$ let δ_ε be a gauge on I such that if $\dot{P} << \delta_\varepsilon$, then $(5.\alpha)$ $\left|S(f;\dot{P}) - \int_I f\right| \le \varepsilon.$

If $\dot{P}_0 = \{(J_j, t_j) : j = 1,\ldots,s\}$ is any δ_ε-fine subpartition of I, then

$(5.\beta)$

$$\left|\sum_{j=1}^{s}\left\{f(t_j)l(J_j) - \int_{J_j} f\right\}\right| = \left|S(f;\dot{P}_0) - \int_{U(P_0)} f\right| \le \varepsilon.$$

Proof. Let $K_1,\ldots.,K_m$ denote closed subintervals in I such that $\{J_j\} \cup \{K_k\}$ form a partition of I. Our basic strategy is to use the fact that f is integrable on each of the intervals $K_1,\ldots.,K_m$ and obtain partitions of these intervals that are so fine that the sum of their contributions is arbitrarily small.

Now let $\alpha > 0$ be arbitrary. Since (by Corollary 3.8) the restriction of f to each subinterval $K_k\,(k = 1,\ldots.,m)$ is integrable, there exists a gauge $\delta_{\alpha,k}$ on K_k such that if $\dot{Q}_k$ is a $\delta_{\alpha,k}$ – fine partition of K_k, then

$(5.\gamma)$ $\left|S(f;\dot{Q}_k) - \int_{K_k} f\right| \le \alpha / m.$

Clearly we may assume that $\delta_{\alpha,k}(x) \le \delta_\varepsilon(x)$ for all $x \in K_k$. Now let $\dot{P}^*$ denote the tagged partition $\dot{P}^* := \dot{P}_0 \cup \dot{Q}_1 \cup \ldots.. \cup \dot{Q}_m$ of I. Evidently $\dot{P}^*$ is δ_ε -fine, so that inequality $(5.\alpha)$ hold for $\dot{P}^*$. Further, it is clear that

$$S(f;\dot{P}^*) = S(f;\dot{P}_0) + S(f;\dot{Q}_1) + \ldots.. + S(f;\dot{Q}_m),$$

$$\int_I f = \int_{U(\dot{P}_0)} f + \int_{K_1} f + \ldots. + \int_{K_m} f.$$

Consequently, we obtain

$$\left|S(f;\dot{P}_0) - \int_{U(\dot{P}_0)} f\right|$$

$$= \left|\left\{S(f;\dot{P}^*) - \sum_{k=1}^{m} S(f;\dot{Q}_k)\right\} - \left\{\int_I f - \sum_{k=1}^{m}\int_{K_k} f\right\}\right|$$

$$\leq \left| S\left(f;\dot{P}^*\right) - \int_I f \right| + \sum_{k=1}^{m} \left| S\left(f;\dot{Q}^*\right) - \int_{K_k} f \right|.$$

If we use inequalities $(5.\alpha)$ and $(5.\gamma)$ this last sum is dominated by

$$\varepsilon + m(\alpha / m) = \varepsilon + \alpha.$$

Since $\alpha > 0$ is arbitrary, then $\left| S\left(f;\dot{P}_0\right) - \int_{U(\dot{P}_0)} f \right| \leq \varepsilon,$ as claimed. Q.E.D.

We now snow that we can interchange the absolute values and the sum in $(5.\beta)$ if we double the possible error.

6.4. Corollary. With the hypotheses of Lemma 6.3, we have

$$(5.\delta) \qquad \sum_{j=1}^{s} \left| f\left(t_j\right) l\left(J_j\right) - \int_{J_j} f \right| \leq 2\varepsilon.$$

Proof. Let $\dot{P}_0^+$ be those pairs in $\dot{P}_0$ for which $f\left(t_j\right) l\left(J_j\right) - \int_{J_i} f \geq 0$, and let $\dot{P}_0^-$ be those pairs for which these terms are <0. Now apply the Saks-Henstock Lemma to both $\dot{P}_0^+$ and $\dot{P}_0^-$. We obtain the inequalities

$$\sum_{\dot{P}_0^+} \left| f\left(t_j\right) l\left(J_j\right) - \int_{J_j} f \right| = \sum_{\dot{P}_0^+} \left\{ f\left(t_j\right) l\left(J_j\right) - \int_{J_j} f \right\} \leq \varepsilon,$$

$$\sum_{\dot{P}_0^-} \left| f\left(t_j\right) l\left(J_j\right) - \int_{J_j} f \right| = -\sum_{\dot{P}_0^-} \left\{ f\left(t_j\right) l\left(J_j\right) - \int_{J_j} f \right\} \leq \varepsilon.$$

If we add these two terms, then we obtain $(5.\delta)$. the following result will be used in Section 7.

6.5. Corollary. With the hypotheses of the Saks-Henstock Lemma 6.3, we have

$$(5.\varepsilon) \qquad \left| \sum_{j=1}^{s} \left| f\left(t_j\right) \right| l\left(J_j\right) - \sum_{j=1}^{s} \left| \int_{J_j} f \right| \right| \leq 2\varepsilon.$$

Proof. One consequence of the Triangle Inequality is that

$$-|A - B| \leq |A| - |B| \leq |A - B|.$$

Notes

If we take $A := f(t_j)l(J_j)$ and $B := \int_{J_j} f$, sum from $j = 1,....,s$, and apply inequality $(5.\delta)$, then we obtain $(5.\varepsilon)$.

6.4 CONTINUITY OF THE INDEFINITE INTEGRALS

We now give an important application of the Saks-Henstock Lemma: we will establish the continuity of the indefinite integrals of an integrable function (stated without proof in 4.11). For simplicity we consider here only the indefinite integral with the left endpoint a as base point, since any other indefinite integral differs from this one by a constant function.

6.6. Theorem. If f belongs to $R^*([a,b])$, then the indefinite integral $F(x) := \int_a^x f$ is continuous on $[a,b]$.

Proof. Let $c \in [a,b)$; we will show that F is continuous from the right at c. If $\varepsilon > 0$, let the gauge δ_ε on $I := [a,b]$ be as in the hypothesis of the Saks-Henstock Lemma 6.3. We now define a gauge by

$$\delta_\varepsilon^{'}(t) := \begin{cases} \min\left\{\delta_\varepsilon(t), \frac{1}{2}|t-c|\right\} & if\ t \in I, t \neq c, \\ \min\left\{\delta_\varepsilon(c), \varepsilon/\left(|f(c)|+1\right)\right\} & if\ t = c. \end{cases}$$

Now let $0 < h < \delta_\varepsilon^{'}(c)$ and let $\dot{P}_0$ be the $\delta_\varepsilon^{'}-$fine subpartition consisting of the single pair $([c, c+h], c)$. If we apply the Saks-Henstock Lemma to $\dot{P}_0$, we infer that

$$\left|f(c)h - \int_c^{c+h} f\right| \leq \varepsilon.$$

Hence it follows from the fact that $h \leq \varepsilon/\left(|f(c)|+1\right)$ that

$$|F(c+h) - F(c)| = \left|\int_c^{c+h} f\right| \leq |f(c)|h + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the F is continuous from the right at c. In the same way we show that F is continuous from the left at any point in $(a,b]$.

The Vitali Covering theorem

In order to give a proof of the differentiation part of the Fundamental. Theorem 4.11 we require a version of the Vitali Covering Theorem, which we do not assume to be known to the reader. Therefore a slight detour will be needed.

6.7. Definition. Let $E \subseteq [a,b]$ and let F be a collection of nondegenerate closed subintervals in $[a-1,b+1]$. We say that F is a Vitali convering for E if for every $x \in E$ and every $s > 0$ there exists an interval $J \in F$ such that $x \in J$ and $0 < l(J) < s$.

It is clear that if F is a Vitali covering for E, then every point $x \in E$ belongs to infinitely many intervals in F. As an example of a countable Vitali covering for the interval $I := [0,1]$, consider the collection of all closed balls $B[r;1/n]$, where $r \in I \cap R$ and $n \in R$.

6.8. Vitali Covering Theorem.: Let $E \subseteq [a,b]$ and let F be a vitali covering for E. Then given $\varepsilon > 0$ there exist disjoint intervals $I_1, \ldots\ldots, I_p$ from F and a countable collection of closed intervals $\{J_i : i = p+1, \ldots..\}$ in with

$$(5.\zeta) \qquad E - \bigcup_{i=1}^{p} I_i \subseteq \bigcup_{i=p+1}^{\infty} J_i \quad \text{And}$$

$$\sum_{i=p+1}^{\infty} l(J_i) \le \varepsilon.$$

Therefore, it follows that

$$(5.\eta) \qquad E \subseteq \bigcup_{i=1}^{p} I_i \cup \bigcup_{i=p+1}^{\infty} J_i.$$

Proof. We choose $I_1 \in F$ arbitrarily and suppose that disjoint intervals $I_1, \ldots, I_r$ from F have already been chosen. If $E \subseteq \bigcup_{i=1}^{r} I_i$, we can take

$J_i = \emptyset$ for $i \geq r+1$ and the proof is complete. Otherwise, we let F_r be the collection of all intervals $I \in F$ that contain points of E and are disjoint from each of the intervals $I_1, \ldots., I_r$. We let $\lambda_r \left(\leq b-a+2\right)$ be the supremum of the lengths of all such intervals $I \in F_r$, and we choose $I_{r+1} \in F_r$ such that $l\left(I_{r+1}\right) > \frac{1}{2}\lambda_r$. This construction gives an infinite sequence of intervals $\left(I_i\right)$, unless E is contained in the union of some finite number of these closed intervals.

Suppose that we obtain an infinite sequence $\left(I_i\right)$. Since the intervals I_i are pairwise disjoint and are contained in the interval $\left[a-1, b+1\right]$, we must have $\sum_{i=1}^{\infty} l\left(I_i\right) \leq b-a+2$ (see Exercise 6.J). Therefore, given $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that we have $\sum_{i=p+1}^{\infty} l\left(I_i\right) \leq \varepsilon/5$. Now let $D_p := E - \bigcup_{i=1}^{p} I_i$ and let $x \in D_p$ be arbitrary. Since F is a vitali covering for E there exists an interval $I_x \in F$ such that $x \in I_x$ and $I_x \cap I_i = \emptyset$ for all $i = 1, \ldots, p$; therefore $I_x \in F_p$. We claim that the interval I_x must intersect at least one interval I_n with $n > p$. For, if $I_x \cap I_i = \emptyset$ for $i = 1, \ldots, n$, then $I_x \in F_n$ and we have $0 < l\left(I_x\right) \leq \lambda_n$. But $0 < \lambda_n < 2l\left(I_{n+1}\right)$, so that $\lim_n \lambda_n = 0$; therefore $0 < l\left(I_x\right) \leq \lambda_n$ cannot hold for all $n \in \mathbb{N}$. Hence let $n\left(x\right) \in \mathbb{N}$ be the smallest integer n such that $I_x \cap I_n \neq \emptyset$, so that $n\left(x\right) > p$ and, since $I_x \in F_{n(x)-1}$, we have $l\left(I_x\right) \leq \lambda_{n(x)-1} < 2l\left(I_{n(x)}\right)$. Since I_x contains the pint $x \in D_p$ and has a point in $I_{n(x)}$, the distance form x to the midpoint $x_{n(x)}$ of $I_{n(x)}$ is $\leq l\left(I_x\right) + \frac{1}{2} l\left(I_{n(x)}\right) < \frac{5}{2} l\left(I_{n(x)}\right)$. Therefore, x belongs to the interval $J_{n(x)}$ with the same midpoint $x_{n(x)}$ as $I_{n(x)}$ and 6 times its length. For $i \geq p+1$, let J_i be formed from I_i in this way. Since $x \in D_p$ is arbitrary, the argument just given implies that

$$(5.\theta) \qquad E - \bigcup_{i=1}^{p} I_i = D_p \subseteq \bigcup_{i=p+1}^{\infty} J_i.$$

Also, since $l(J_i) = 5l(I_i)$ for $i > p,$ we have $\sum_{i=p+1}^{\infty} l(J_i) \le \varepsilon.$ Q.E.D.

The Differentiation Theorem

We are finally prepared to prove the difficult part of Theorem 4.11.

6.9 Differentiation Theorem. Let f be integrable on $I := [a,b]$ and let F be an indefinite integral of f. Then there exists a null set $Z \subset I$ such that if $x \in I - Z$ then $F'(x)$ exists and equals $f(x)$; thus, F is an a-primitive of f.

Proof. As usual, it is enough to handle the indefinite integral F of f with base point a.

We let E be the set of points $x \in [a,b)$ such that the right hand derivative $F_+'(x)$ of F either does not exist at x or does not equal $f(x)$. We will show that E is a null set, and a similar argument shows that the set of points in $(a,b]$ where F does not have a left hand derivative equal to $f(x)$ also is a null set. Since the set Z of points where F does not have a derivative equal to $f(x)$ is the union of these two sets, the set Z is a null set.

If F has a right hand derivative $F_+'(x) = f(x)$ at the point $x \in I,$ then for any $\alpha > 0$ there exists an $s > 0$ such that if $v \in I$ is any number with $x < v < x + s,$ then

$$\left| \frac{F(v) - F(x)}{v - x} - f(x) \right| \le \alpha.$$

Negating this assertion, if $x \in E,$ then there exists $\alpha(x) > 0$ such that for every $s > 0$ there exists a point $v_{x,s} \in I$ with $x < v_{x,s} < x + s$ and such that

$$(5.l) \qquad \left| \frac{F(v_{x,s}) - F(x)}{v_{x,s} - x} - f(x) \right| > \alpha(x),$$

Whence it follows that

$$(5.k) \qquad \left\|\left[F\left(v_{x,s}\right)-F\left(x\right)\right]-f\left(x\right)\left(v_{x,s}-x\right)\right|>\alpha\left(x\right)\left(v_{x,s}-x\right).$$

Fix $n \in R$ and let $E_n := \{x \in E : \alpha(x) \geq 1/n\}$. Given $\varepsilon > 0$, since f is integrable, there exists a gauge δ_ε on I such that if $\dot{P}$ is a $\delta_\varepsilon -$fine partition of I, then

$$(5.\lambda) \qquad \left|S\left(f;\dot{P}\right)-\int_1 f\right| \leq \varepsilon/n.$$

Now let $F_n := \{[x, v_{x,s}] : x \in E_n, 0 < s \leq \delta_\varepsilon(x)\}$; then F_n is a Vitali covering for E_n. By the Vitali Covering Theorem there exist intervals $I_1 := [x_1, v_1], ..., I_p := [x_p, v_p]$ in F_n and a sequence $(J_i)_{p+1}^{\infty}$ of closed intervals such that

$$(5.\mu) \qquad E_n \subseteq \bigcup_{i=1}^{p} I_i \cup \bigcup_{i=p+1}^{\infty} J_i \qquad \text{and}$$

$$\sum_{i=p+1}^{\infty} l(J_i) \leq \varepsilon.$$

We now consider the sum

$$(5.v)$$

$$\sum_{i=1}^{p}\left|f\left(x_i\right)\left(v_i-x_i\right)-\int_{x_i}^{v_i} f\right|=\sum_{i=1}^{p}\left|f\left(x_i\right)\left(v_i-x_i\right)-\left[F\left(v_i\right)-F\left(x_i\right)\right]\right|.$$

It follows from $(5.k)$ with $\alpha(x_i) \geq 1/n$ that the sum on the right in $(5.v)$ is greater than

$$(5.\xi) \qquad (1/n)\sum_{i=1}^{p}(v_i - x_i).$$

On the other hand, since $x_i \leq v_i \leq x_i + \delta_\varepsilon(x_i)$ for $i = 1, \ldots, p$, the ordered pairs $\{(I_i, x_i)\}_{i=1}^{p}$ form a subpartition of a $\delta_\varepsilon -$fine partition of I for which $(5.\lambda)$ holds. Therefore, by Corollarly 6.4 of the Saks-Henstock Lemma, we conclude that the sum in $(5.v)$ is less than or equal to $2\varepsilon / n$. If we combine this estimate with $(5.\xi)$, we find (after multiplying by n) that

$$(5.o) \qquad \sum_{i=1}^{p}(\upsilon_i - x_i) \le n\sum_{i=1}^{p}\left| f(x_i)(\upsilon_i - x_i) - \int_{x_i}^{\upsilon_i} f \right| \le 2\varepsilon.$$

But, in view of $(5.\mu)$, we conclude that E_n is contained in a countable in a countable union of intervals with total length $\le 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that the set E_n is a null set. Therefore, since $E = \bigcup_{n=1}^{\infty} E_n$ and each E_n is a null set, we conclude that E is a null set.

Theorem 4.11 has now been completely proved.

6.5 CHARACTERIZATION OF NULL FUNCTIONS

We now establish some equivalent properties for a function to be a null function in the sense of Definition 2.4(b).

6.10 Characterization of null functions. Let $\psi : I \to \mathbb{R}$ where $I := [a,b]$. Then the following statements are equivalent:

(a) $|\psi|$ is a null function on I.

(b) ψ is a null function on I.

(c) $\psi \in R^*(I)$ and $\int_a^r \psi = 0$ for every rational $r \in \mathbb{Q} \cap I$.

(d) An indefinite integral $\Psi_c(x) := \int_c^x \psi \ of \ \psi$ vanishes identically on I.

(e) $|\psi| \in R^*(I)$ and $\int_a^b |\psi| = 0$.

(f) $|\psi| \in R^*(I)$ and $\int_a^b |\psi| = 0$ for all $x \in I$.

Proof. $(a) \Leftrightarrow (b)$ Since $\{x \in I : |\psi(x)| \ne 0\} = \{x \in I : \psi(x) \ne 0\}$, It follows that $|\psi|$ is a null function if and only if ψ is a null function.

$(b) \Rightarrow (c)$ If ψ is a null function, example 2.6 (b) implies that ψ is in $R^*(I)$. By Corollarly 3.8, the restriction of ψ to every subinterval $[a,r]$ is integrable. Since these restrictions are null functions, Example 2.6(b) implies that $\int_a^r \psi = 0.$

$(c) \Rightarrow (d)$ if $\psi \in R^*(I)$ and $\Psi_c(x) = \int_c^x \psi,$ then since $\Psi_c(r) = 0$ for all $r \in \mathbb{Q} \cap I,$ it follows from the continuity of Ψ_c that it vanishes identically on I.

$(d) \Rightarrow (b)$ if the indefinite integral Ψ_c of ψ vanishes identically, then $\Psi_c^{'}(x) = 0$ for all $x \in I.$ By the Differentiation Theorem 6.9, we conclude that $\psi(x) = \Psi_c^{'}(x) = 0$ except for x in some null set. Therefore, ψ is a null function.

$(a) \Rightarrow (e)$ This assertion follows form Example 2.6 (b) applied to $|\psi|$.

$(e) \Rightarrow (f)$ if $|\psi| \in R^*(I)$ and $\int_a^b |\psi| = 0,$ then Theorem 3.7 implies that the restrictions of $|\psi|$ to $[a,x]$ and $[x,b]$ are integrable. From Theorem 3.2, we have

$$0\int_a^b |\psi| = \int_a^x |\psi| + \int_x^b |\psi| \geq \int_a^x |\psi| \geq 0.$$

Therefore $\int_a^x |\psi| = 0$ for all $x \in I.$

$(f) \Rightarrow (a)$ The hypothesis is that the indefinite integral of $|\psi|$ with base point a vanishes identically, whence it follows from the Differentiation Theorem 6.9 that $|\psi|$ is a null function.

A characterization of Indefinite Integrals

We conclude this section by giving a necessary and sufficient condition that a function F be an indefinite integral of a function $f \in R^*([a,b])$. The reader will see that the proof of the second part of this theorem is essentially the same as the proofs of the Fundamental Theorems 4.6 and 4.7.

The following notion was explicitly formulated by Vyborny [V;p.427]. It uses the notion of (δ, E)-fineness of a subpartition given in Definition 6.1 (d).

6.11 Definition. A function $F : I \to \mathbb{R}$ is said to have negligible variation on a set $E \subseteq I$ and we write $F \in NV_1(E)$ if, for every $\varepsilon > 0$ there exists a gauge δ_ε on E such that if $\dot{P}_0 := \{([u_j, v_j], t_j)\}_{j=1}^{s}$ is any (δ_ε, E)-fine subpartition of I, then

$$(5.\pi) \qquad \sum_{j=1}^{s} \left| F(v_j) - F(u_j) \right| \le \varepsilon.$$

It will be seen in the exercises that if $F \in NV_1(E)$, then F is continuous at every point of E. Conversely, if C is a countable set in I and $F : I \to \mathbb{R}$ is continuous at every point of C, then $F \in NV_1(C)$. However, when $Z \subset I$ is a null set. Not every continuous function on I belongs to $NV_1(Z)$. For example, the Cantor-Lebesgue singular function $\Lambda : [0,1] \to \mathbb{R}$, introduced in 4.17, is monotone and continuous on $[0,1]$, but it is not in $NV_{[0,1]}(\Gamma)$.

6.12 Characterization Theorem. A function $G : I \to \mathbb{R}$ is an indefinite integral of a function $f \in R^*(I)$ if any only if there exists a null set $Z \subset I := [a,b]$ such that $G'(x) = f(x)$ for all $x \in I - Z$ and $G \in NV_I(Z)$. In this case, we have

$$(5.\rho) \qquad \int_a^x f = G(x) - G(a) \qquad \text{for all} \qquad x \in I.$$

Proof. $(\Rightarrow)$ If $f \in R^*(I)$ and $F(x) := \int_a^x f$, then it follows from the Differentiation Theorem 6.9 that there exists a null set $Z \subset I$ such that $F'(x) = f(x)$ for all $x \in I - Z$. We define f_1 on I by $f_1(x) := f(x)$ for $x \in I - Z$ and $f_1(x) := 0$ for $x \in Z$. It follows from Exercise 3.C that $f_1 \in R^*(I)$ and that F is also the indefinite integral of f_1 with base point a. Therefore, given $\varepsilon > 0$, there exists a gauge η_ε on I such that if $\dot{P} << \eta_\varepsilon$, then

$$\left|\int_a^b f_1 - S\left(f_1;\dot{P}\right)\right| \le \frac{1}{2}\varepsilon.$$

Now let $\dot{P}_0 := \left\{\left(\left[u_j,v_j\right],t_j\right)\right\}_{j=1}^s$ be any $\left(\eta_\varepsilon, Z\right)$-fine subpartition of I. Then $\dot{P}_0$ is a subset of some η_ε-fine partition $\dot{P}$ of I. It follows from Corollary 6.4 that

$$\sum_{j=1}^{s}\left|f_1\left(t_j\right)\left(v_j - u_j\right) - \int_{u_j}^{v_j} f_1\right| \le \varepsilon.$$

But since $f_1\left(t_j\right) = 0$ and $\int_{u_j}^{u_j} f_1 = F\left(v_j\right) - F\left(u_j\right)$ for $j = 1,\ldots.,s,$ we conclude that $\sum_{j=1}^{s}\left|F\left(v_j\right) - F\left(u_j\right)\right| \le \varepsilon.$ Since $\dot{P}_0$ is an arbitrary $\left(\eta_\varepsilon, Z\right)$-fine subpartition of I, we infer that F has negligible variation on Z.

If G is any indefinite integral of f, then $G = F + G\left(a\right)$ so that $G^{'}\left(x\right) = f\left(x\right)$ for all $x \in I - Z,$ and since $G\left(v_j\right) - G\left(u_j\right) = F\left(v_j\right) - F\left(u_j\right),$ it follows that $G \in NV_1\left(Z\right).$ Also, $F\left(x\right) = G\left(x\right) - G\left(a\right)$ so that equation $\left(5.\rho\right)$ results.

$\left(\Leftarrow\right)$ Suppose that Z is a null set and that $G \in NV_1\left(Z\right)$ is differentiable on I-Z. We define $f\left(x\right) := G^{'}\left(x\right)$ for $x \in I - Z$ and $f\left(x\right) := 0$ for $x \in Z.$ We will show that $f \in R^*\left(I\right)$ and that G is an indefinite integral of f.

Given $\varepsilon > 0,$ we will construct a gauge for f. If $t \in I - Z,$ choose $\delta_\varepsilon\left(t\right) > 0$ as in the Straddle Lemma 4.4 such that if $t \in \left[u,v\right] \subseteq \left[t - \delta_\varepsilon\left(t\right), t + \delta_\varepsilon\left(t\right)\right],$ then

$$\left|G\left(v\right) - G\left(u\right) - f\left(t\right)\left(v - u\right)\right| \le \varepsilon\left(v - u\right).$$

For $t \in Z,$ choose $\delta_\varepsilon(t) > 0$ as required in Definition 6.11. Thus we have defined a gauge δ_ε on I.

We now let $\dot{P} := \{([u_i, v_i], t_i)\}_{i=1}^{n}$ be a δ_ε-fine partition of I. Using the telescoping sum $G(b) - G(a) = \sum_{i=1}^{n}[G(v_i) - G(u_i)],$ it is readily seen that

$$\left|G(b) - G(a) - S(f;\dot{P})\right| = \left|\sum_{i=1}^{n}\left[G(v_i) - G(u_i) - f(t_i)(v_i - u_i)\right]\right|.$$

We break this sum into sums over terms where $t_i \in Z$ (where $f(t_i) = 0,$ and over terms where $t_i \in I - Z.$ We conclude that this sum is

$$\leq \sum_{t_i \in Z}\left|G(v_i) - G(u_i)\right| + \sum_{t_i \in I-Z}\left|G(v_i) - G(u_i) - f(t_i)(v_i - u_i)\right|$$

$$\leq \varepsilon + \sum_{t_i \in I-Z}\varepsilon(v_i - u_i) \leq \varepsilon(1 + b - a).$$

Since $\varepsilon > 0$ is arbitrary, then $f \in R^*(I)$ and $G(b) - G(a) = \int_a^b f.$ Since the same argument can be applied to any interval $[a, x] \subseteq I,$ the assertion is proved.

This theorem can be readily used to show that every L-integrable function on I belongs to $R^*(I),$ provided that we known that the indefinite integral of an L-integrable function is “absolutely continuous”. (See Section 14.) Another characterization of the indefinite integral of functions in $R^*(I)$ is given in the book of Gordon [G-3; p.147], and will be mentioned in Section 14.

Check your progress

1. Explain the thermo of Continuity of the Indefinite Integrals

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2. Explain the The Vitali Covering theorem

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3. Explain the Characterization of Null Functions

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6.6 LET US SUM UP

1. Let $I := [a,b]$ be a nondegenerate compact interval.

(a) A subpartition of I is a collection $\{J_j\}_{j=1}^{s}$ of nonoverlapping closed intervals in I.

(b) A tagged subpartition of I is a collection $\dot{P}_0 = \{(J_j, t_j)\}_{j=1}^{s}$ of ordered pairs, consisting of intervals $\dot{P}_0 = \{J_j\}_{j=1}^{s}$ that form a subpartition of I, and tags $t_j \in J_j$ for $j = 1,\ldots\ldots, s.$

2. Let $E \subseteq [a,b]$ and let F be a collection of nondegenerate closed subintervals in $[a-1, b+1]$. We say that F is a Vitali convering for E if for every $x \in E$ and every $s > 0$ there exists an interval $J \in F$ such that $x \in J$ and $0 < l(J) < s.$

3. A function $G : I \to R$ is an indefinite integral of a function $f \in R^*(I)$ if any only if there exists a null set $Z \subset I := [a,b]$ such that $G^{'}(x) = f(x)$ for all $x \in I - Z$ and $G \in NV_I(Z)$. In this case, we have $(5.\rho)$ $\int_a^x f = G(x) - G(a)$ for all $x \in I.$

6.7 KEY WORDS

Sakes –Henstock lemma

Indefinite integrals

Null function

6.8 QUESTIONS FOR REVIEW

1. Explain about The Sakes- Henstock Lemma

2. Explain about continuity of the indefinite integrals

3. Explain about characterization of null functions

6.9 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

6.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 6.4

2. See section 6.4

3. See section 6.5

UNIT-7 MEASURABLE FUNCTIONS

STRUCTURE

7.0 Objective

7.1 Introduction

7.2 Definitions related to Measurable functions

7.3 Complex Valued functions

7.4 Functions between measurable spaces

7.5 Let us sum up

7.6 Key words

7.7 Questions for review

7.8 Suggestive readings and references

7.9 Answers to check your progress

7.0 OBJECTIVE

this unit we will learn and understand about definitions related to measurable functions, Complex valued functions, Functions between measurable spaces.

7.1 INTRODUCTION

In developing the Lebesgue integral we shall be concerned with classes of real-valued functions defined on a set X. In various applications the set X may be the unit interval $I = [0,1]$ consisting of all real numbers x satisfying $0 \le x \le 1$; it may be the set $N = \{1, 2, 3, \ldots\}$ of natural numbers; it may be the entire real line R; it may be all of the plane; or it may be some other set. Since the development of the integral does not depend on the character of the underlying space X, we shall make no assumptions about its specific nature.

Given the set X, we single out the family X of subsets of X which are "Well-behaved" in a certain technical sense. To be precise, we shall assume that this family contains the empty set ϕ and the entire set X, and that X is closed under complementation and countable unions.

7.2 DEFINITIONS RELATED TO MEASURABLE FUNCTIONS

A family X of subsets of a set X is said to be a $\sigma-$algebra (or a $\sigma-$ field) in case:

(i) ϕ, X belong to X.

(ii) If A belong to X, then the complement $\wp(\mathrm{A}) = \mathrm{X}/\mathrm{A}$ belongs to X.

(iii) If (A_n) is a sequence of sets in X, then the union $\cup_{n=1}^{\infty}\mathrm{A}_n$ belongs to X.

An ordered pair $\cup_{n=1}^{\infty}\mathrm{A}_n$ consisting of a set X and a $\cup_{n=1}^{\infty}\mathrm{A}_n$ $\sigma-$algebra X of subsets of X is called a measurable space. Any set in X is called an X-measurable set, but when the $\sigma-$algebra X is fixed (as is generally the case), the set will usually be said to be measurable.

The reader will recall the rules of De Morgan:

(7.1) $$\wp\left(\underset{\alpha}{\cup}\mathrm{A}_\alpha\right) = \underset{\alpha}{\cap}\wp(\mathrm{A}_\alpha), \quad \wp\left(\underset{\alpha}{\cap}\mathrm{A}_\alpha\right) = \underset{\alpha}{\cup}\wp(\mathrm{A}_\alpha).$$

It follows from these that the intersection of a sequence of sets in X also belongs to X.

We shall now give some examples of $\sigma-$algebras of subsets.

7.2 EXAMPLES. (a) Let X be any set and let X be the family of all subsets of X.

(b) Let X be the family consisting of precisely two subsets of X, namely ϕ and X.

(c) Let $X = \{1, 2, 3, \ldots\}$ be the set N of natural numbers and let X consist of the subsets

$$\phi, \{1, 3, 5, \ldots\}, \quad \{2, 4, 6, \ldots\}, \quad X.$$

(d) Let X be an uncountable set and x be the collection of subsets which are either countable or have countable complements.

(e) If X_1 and X_2 are $\sigma-$algebras of subsets of X, let X_3 be the intersection of X_1 and X_2. It is readily checked that X_3 is a $\sigma-$algebra.

(f) Let A be a nonempty collection of subsets of X. We observe that there is a smallest $\sigma-$algebra of subsets of X containing A. To see this, observe that the family of all subsets of X is a $\sigma-$algebra containing A and the intersection of all the $\sigma-$algebras containing A. This smallest $\sigma-$algebra is sometimes called the $\sigma-$algebra generated by A.

(g) Let X be the set R of real numbers. The Borel algebra is the $\sigma-$algebra B generated by all open intervals (a,b) in R. Observe that the Borel algebra B is also the $\sigma-$algebra generated by all closed intervals $[a, b]$ in R. Any set in B is called a Borel set.

(h0 Let X be the set $\bar{R}$ of extended real numbers. If E is a Borel subset of R, let

(7.2) $$E_1 = E \cup \{-\infty\}, \qquad E_2 = E \cup \{+\infty\},$$

$E_3 = E \cup \{-\infty, +\infty\}$, and let $\bar{B}$ be the collection of all sets E, E_1, E_2, E_3 as E varies over B. It is readily seen that $\bar{B}$ is a $\sigma-$algebra and it will be called the extended Borel algebra.

In the following, we shall consider a fixed measurable space (X, X).

7.3 DEFINITION. A function f on X to R is said to be X-measurable (or simply measurable) if for every real number α the set

(7.3) $\{x \in X : f(x) < \alpha\}$

Belongs to X. The next lemma shows that we could have modified the form of thesets in defining measurability.

7.4 LEMMA. The following statements are equivalent for a function f on X to R:

(a) For every $\alpha \in R$, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$ belongs to X.

(b) For every $\alpha \in R$, the set $B_\alpha = \{x \in X : f(x) \leq \alpha\}$ belongs to X.

(c) For every $\alpha \in R$, the set $C_\alpha = \{x \in X : f(x) \geq \alpha\}$ belongs to X.

(d) For every $\alpha \in R$, the set $D_\alpha = \{x \in X : f(x) < \alpha\}$ belongs to X.

PROOF. Since B_α and A_α are complements of each other, statement (a) is equivalent to statement (b). Similarly, statements (c) and (d) are equivalent. If (a) holds, then $A_{\alpha-1/n}$ belongs to x for each n and since

$$C_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha-1/n},$$

It follows that $C_\alpha \in X.$ Hence (a) implies(c). Since

$$A_\alpha = \bigcap_{n=1}^{\infty} C_{\alpha+1/n},$$

It follows that (c) implies (a).

7.5 Examples. (a) Any constant function is measureable. For if $f(x) = c$ for all $x \in X$ and if $\alpha \geq c$, then

$$\{x \in X : f(x) > \alpha\} = \phi,$$

Whereas if $\alpha < c$, then $\{x \in X : f(x) > \alpha\} = x$.

b) If $E \in X$, then the characteristic function X_E, defined by

$$X_g(x) = 1, x \in E,$$
$$= 0, x \notin E,$$

is measurable, In fact, $\{x \in X : x_g(x) > \alpha\}$ is either X, E, or θ.

c) If x is the set **R** of real numbers, and X is the Borel algebra **B**, then any continous function f on **R** is Borel measurable (that is, **B-**measurable). In fact, if f is continuous, then $\{x \in R : f(x) > \alpha\}$ is an open set in **R** and hence is the union of a sequence of open intervals Therefore, it belongs to **B.**

d) If X=R and X=B, then any monotone function is Borel measurable. For, suppose that f is monotone increasing in the sense that $x \le x'$ implies $f(x) \le f(x')$. Then $\{x \in R : f(x) > \alpha\}$ consists of a half-line which is wither of the form $\{x \in R : x > a\}$ or the form $\{x \in R : x \ge a\}$, or is R or θ.

Certain simple algebraic combinations of measurable functions are measurable, as we shall now show.

7.6 LEMMA. Let f and g be measurable real-valued functions and let c be a real number. Then the functions

$$cf,\ f^2,\ f+g,\ fg,\ |f|,$$

Are also measurable.

PROOF: a) If c=0, the statement is trivial. If $c > 0$, then

$\{x \in X : cf(x) > \alpha\} = \{x \in X : f(x) > \alpha / c\} \in X.$

The case $c < 0$ is handled similarly.

(b) If $\alpha < 0$, then $\{x \in X : (f(x))^2 > \alpha\} = X$; if $\alpha \ge 0$, then

$\{x \in X : (f(x))^2 > \alpha\}$

$$\{x \in X : f(x) > \sqrt{\alpha}\} \cup \{x \in X : f(x) < -\sqrt{\alpha}\}.$$

(c) By hypothesis, if r is a rational number, then

$$S_r = \{x \in X : f(x) > r\} \cap \{x \in X : g(x) > \alpha - r\}$$

Belongs to X. Since it is readily seen that

$$\{x \in X : (f+g)(x) > \alpha\} = \cup\{S_r : r \text{ rational}\},$$

It follows that $f+g$ is measurable.

(d) Since $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$, it follows from parts (a), (b), and (c) that fg is measurable.

(e) If $\alpha < 0$, then $\{x \in X : |f(x)| > \alpha\} = X$, whereas if $\alpha \geq 0$, then

$\{x \in X : |f(x)| > \alpha\} = \{x \in X : f(x) > \alpha\} \cup \{x \in X : f(x) < -\alpha\}$.

Thus the function $|f|$ is measurable.

If f is any function on X to R, let f^{+} and f^{-} be the nonnegative functions defined on X by

(7.4) $f^{+}(x) = \sup\{f(x), 0\}, f^{-}(x) = \sup\{-f(x), 0\}$.

The function f^{+} is called the positive part of f and f^{-} is called the negative part of f. It is clear that

(7.5) $f = f^{+} - f^{-}$ and $|f| = f^{+} + f^{-}$

And it follows from these identities that

(7.6) $f^{+} = \frac{1}{2}(|f| + f), f^{-} = \frac{1}{2}(|f| - f)$.

In view of the preceding lemma we infer that f is measurable if and only if f^{+} and f^{-} are measurable.

The preceding discussion pertained to real-valued functions defined on a measurable space. However, in dealing with sequences of measurable functions we often wish to form suprema, limits, etc., and it is technically convenient to allow the extended real numbers $-\infty, +\infty$ to be taken as values. Hence we wish to define measurability for extended real-valued functions and we do this exactly as in Definition 7.3.

7.7 DEFINITION. An extended real-valued function on X is X-measurable in case that set $\{x \in X : f(x) > \alpha\}$ belongs to X for each real number α. The collection of all extended real-valued X-measurable functions on X is denoted by $M(X, X)$.

Observe that if $f \in M(X, X)$, then

$$\{x \in X : f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) > n\},$$

$$\{x \in X : f(x) = -\infty\} = \wp\left[\bigcap_{n=1}^{\infty} \{x \in X : f(x) > -n\}\right],$$

So that both of these sets belong to X.

The following lemma is often useful in treating extended real-valued functions.

7.8 LEMMA. An extended real-valued function f is measurable if and only if the sets

$$A = \{x \in X : f(x) = +\infty\}, B = \{x \in X : f(x) = -\infty\}$$

Belong to X and the real-valued function f_1 defined by

$$f_1(x) = f(x), \quad \text{if } x \notin A \cup B,$$
$$= 0, \quad \text{if } x \in A \cup B,$$

Is measurable.

PROOF. If f is in $M(X, X)$, it has already been noted that A and B belong to X. Let $\alpha \in R$ and $\alpha \geq 0$, then

$$\{x \in X : f_1(x) > \alpha\} = \{x \in X : f(x) > \alpha\} / A.$$

If $\alpha < 0$, then

$$\{x \in X : f_1(x) > \alpha\} = \{x \in X : f(x) > \alpha\} \cup B.$$

Hence f_1 is measurable.

Conversely, if $A, B \in X$ and f_1 is measurable, then

$$\{x \in X : f(x) > \alpha\} = \{x \in X : f_1(x) > \alpha\} \cup A$$

When $\alpha \geq 0$, and

$$\{x \in X : f(x) > \alpha\} = \{x \in X : f_1(x) > \alpha\} / B$$

When $\alpha < 0$. Therefore f is measurable.

It is a consequence of Lemma 7.6 and 7.8 that if f is in $M(X, X)$, then the functions

$$cf, f^2, |f|, f^+, f^-$$

Also belong to $M(X, X)$.

The only comment that need be made is that we adopt the convention that $0(\pm\infty) = 0$ so that cf vanished identically when C=0. If f and g belong to $M(X, X)$, then the sum $f + g$ is not well-denined by the formula $(f+g)(x) = f(x) + g(x)$ on the sets.

$$E_1 = \{x \in X : f(x) = -\infty \text{ and } g(x) = +\infty\},$$
$$E_2 = \{x \in X : f(x) = +\infty \text{ and } g(x) = -\infty\},$$

Both of which belong to X. However, if we define $f+g$ to be zero on $E_1 \cup E_2$, the resulting function on X is measurable. We shall return to the measurability of the product fg after the next result.

7.9 LEMMA. Let (f_n) be a sequence in $M(X,X)$ and define the functions

$$f(x) = \inf f_n(x), \quad F(x) = \sup f_n(x),$$
$$f^*(x) = \lim\inf f_n(x), \quad F^*(x) = \lim\sup f_n(x).$$

Then $f, F, f^*, \text{and } F^*$ belong to $M(X,X)$.

PROOF. Observe that

$$\{x \in X : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) \geq \alpha\},$$
$$\{x \in X : F(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) > \alpha\},$$

So that f and F are measurable when all the f_n are. Since

$$f^*(x) = \sup_{n\geq 1}\left\{\inf_{m\geq n} f_m(x)\right\},$$
$$F^*(x) = \inf_{n\geq 1}\left\{\sup_{m\geq n} f_m(x)\right\},$$

The measurability of $f^* \text{ and } F^*$ is also established.

7.10 COROLLARY. If (f_n) is a sequence in $M(X,X)$ which converges to f on X, then f is in $M(X,X)$.

PROOF. In this case $f(x) = \lim f_n(x) = \lim\inf f_n(x)$.

We now return to the measurability of the product f g when f, g belong to $M(X,X)$. If $n \in N$, let f_n be the "truncation of f" defined by

$$\begin{aligned} f_n(x) &= f(x), && \text{if } |f(x)| \leq n, \\ &= n, && \text{if } f(x) > n, \\ &= -n, && \text{if } f(x) < -n. \end{aligned}$$

Let g_m be defined similarly. It is readily seen that $f_n \text{ and } g_m$ are measurable (see Exercise 7.k). It follows from Lemma 7.6 that the product $f_n\, g_m$ is measurable. Since

Notes

$f(x)g_m(x) = \lim_n f_n(x)g_m(x), \ x \in X,$ it follows from Corollary 7.10 that fg_m belongs to $M(X, X)$. Since

$$(fg)(x) = f(x)g(x) = \lim_m f(x)g_m(x), \ x \in X,$$

Another application of Corollary 7.10 shows that f g belongs to $M(X, X)$.

It has beeb seen that the limit of a sequence of functions in $M(X, X)$ belongs to $M(X, X)$. We shall now prove that a nonnegative function f in $M(X, X)$ is the limit of a monotone increasing sequence (φ_n) in $M(X, X)$. Moreover, each φ_n can be chosen to be nonnegative and to assume only a finite number of real values.

7.11 LEMMA. If f is a nonnegative function in $M(X, X)$, then there exists a sequence (φ_n) in $M(X, X)$ such that

(a) $0 \le \varphi_n(x) \le \varphi_{n+1}(x)$ for $x \in X, n \in N.$

(b) $0 \le \varphi_n(x) \le \varphi_{n+1}(x)$ for $x \in X, n \in N.$

(c) Each φ_n has only a finite number of real values.

PROOF. Let n be a fixed natural number. If $K = 0, 1, \ldots, n2^n - 1,$ let E_{kn} be the set

$$E_{kn} = \{x \in X : k2^{-n} \le f(x) < (k+1)2^{-n}\},$$

And if $k = n2^n,$ let E_{kn} be the set $\{x \in X : f(x) \ge n\}$. We observe that the sets $\{E_{kn} : k = 0, 1, \ldots, n2^n\}$ are disjoint, belong to X, and have union equal to X. If we define φ_n to be equal to $k2^{-n}$ on E_{kn}, then φ_n belongs to $M(X, X)$. It is reasily established that the properties (a), (b), (c) hold.

7.3 COMPLEX-VALUED FUNCTIONS

It is frequently important to consider complex-valied functions defined on X and to have a notion of measurability for such functions. We observe that if f is a complex-valied function defined on X, then there exist two uniquely determined real-valued functions f_1, f_2 such that

$$f = f_1 + if_2.$$

(Indeed, $f_1(x) = \operatorname{Re} f(x), f_2(x) = \operatorname{Im} f(x), \text{for } x \in X.$) We define the complex-valued function f to be measurable if and only fi its real and imaginary parts f_1 and f_2, respectively, are measurable. It is easy to see that sums, products, and limits of complex-valued measurable functions are also measurable.

7.4 FUNCTIONS BETWEEN MEASURABLE SPACES

In the sequel we shall require the notion of measurability only for real- and complex-valued functions. In some work, however, one wishes to define measurability for a function f from onw measurable space (X, X) into another measurable space (Y, Y). In this case onw says that f is measurable in case the set

$$f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Belong to X for every set E belonging to Y. Although this definition of measurability appears to differ from Definition 7.3, it is not difficult to show (see Exercise 7.P) that Definition 7.3 is equivalent to this definition in the case that $Y = R \text{ and } Y = B.$

This definition of measurability shows very clearly the close analogy between the measurable functions on a measurable space and continuous functions on a topological space.

Exercises :

7.A. show that $[a, b] = \bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n).$ Hence any $\sigma -$ algebra of subsets of R which contains all open intervals also contains that any $\sigma -$algebra containing all closed intervals also contains all open intervals.

7.B. Show that the Borel algebra B is also generated by the collection of all half-open intervals $(a,b] = \{x \in R : a < x \le b\}$. Also show that B is generated by the collection of all half-rays $\{x \in R : x > a\}, a \in R$.

7.C. Let (A_n) be a sequence of subsets of subsets of a set X. Let $E_0 = \theta$ and for $n \in N$, let

$$E_n = \bigcup_{k=1}^{n} A_k, F_n = A_n \setminus E_{n-1}.$$

Show that (E_n) is a monotone increasing sequence of sets and that (F_n) is a disjoint sequence of sets $(\text{that is}, F_n \cap F_m = \theta \text{ if } n \ne m)$ such that

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} A_n.$$

7.D. Let (A_n) be a sequence of subsets of a set X. If A consists of all $x \in X$ which belong to infinitely many of the sets A_n, show that

$$A = \bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} A_n \right].$$

The set A is often called the limit superior of the sets (A_n) and denoted by lim inf A_n.

7.E. Let (A_n) be a sequence of subsets of a set X. If B consists of all $x \in X$ which belong to all but a finite number of the sets $A_{n,}$ show that

$$B = \bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} A_n \right].$$

The set B is often called the limit inferior of the sets (A_n) and denoted by lim inf A_n.

7.F. If (E_n) is a sequence of subsets of a set X which is monotone increasing $(\text{that is}, E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots)$, show that Lim sup $E_n = \bigcup_{n=1}^{\infty} E_n = \text{lim inf } E_n$.

7.G. If (F_n) is a sequence of subsets of a set X which is monotone decreasing $(\text{that is}, F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots)$, show that

Lim sup $F_n = \bigcup_{n=1}^{\infty} F_n = \lim\inf F_n$.

7.H. If (A_n) is a sequence of subsets of X, show that

$$\theta \subseteq \lim\inf A_n \subseteq \lim\sup A_n \subseteq X.$$

Give an example of a sequence (A_n) such that lim inf $A_n = \theta$, lim sup $A_n = X$. Give an example of a sequence (A_n) which is neither monotone increasing or decreasing, but is such that $\lim\inf A_n = \lim\sup A_{n.}$

When this equality holds, the common value is called the limit of (A_n) and is denoted by lim A_n.

7.I. Give an example of a function f on X to R which is not X-measurable, but is such that the functions $|f| \text{ and } f^2$ are X-measureable.

7.J. If a,b,c are real numbers, let mid (a,b,c) denote the "value in the middle." Show that

mid $(a,b,c) = \inf \{\sup\{a,b\}, \sup\{a,c\}, \sup\{b,c\}\}$.

If f_1, f_2, f_3 are X-measurable functions on X to R and if g is defined for $x \in X$ by

$$g(x) = \text{mid}(f_1(x), f_2(x), f_3(x)),$$

Then g is X-measurable.

7.K. show directly (without using the preceding exercise) that if f is measurable and A>0, then the truncation f_A defined by.

$$\begin{aligned} f_A(x) &= f(x), \text{ if } && |f(x)| \le A, \\ &= A, && \text{if } \quad f(x) > A, \\ &= -A, && \text{if } \quad f(x) < -A, \end{aligned}$$

is measurable.

7.L. Let f be a nonnegative X-measurable function on X which is bounded (that ism there exists a constant K such that $0 \le f(x) \le K$ for all x in X). Show that the sequence (ϕ_n) constructed in Lemma 7.11 converges uniformly on X to f.

7.M. Let f be a function defined on a set X with values in a set Y. If E is any subset of Y, let

Notes

$$f^{-1}(E)=\{x\in X:f(x)\in E\}.$$

Show that $f^{-1}(\theta)=\theta, f^{-1}(Y)=X.$ If E and F are subsets of Y, then

$$f^{-1}(E\setminus F)=f^{-1}(E)\setminus f^{-1}(F).$$

If $\{E_\alpha\}$ is any nonempty collection of subsets of Y, then

$$f^{-1}\left(\bigcup_\alpha E_\alpha\right)=\bigcup_\alpha f^{-1}(E_\alpha), f^{-1}\left(\bigcap_\alpha E_\alpha\right)=\bigcap_\alpha f^{-1}(E_\alpha).$$

In particular it follows that if Y is a $\sigma-$algebra of subsets of Y, then $\{f^{-1}(E):E\in Y\}$ is a $\sigma-$algebra of subsets of X.

7.N. Let f be a function defined on a set X with values in a set Y. let X be a $\sigma-$algebra of subsets of X and let $Y=\{E\subseteq Y:f^{-1}(E)\in X\}$. show that Y is a $\sigma-$algebra.

7.O. Let (X,X) be a measurable space and f be defined on X to Y. Let A be a collection of subsets of Y such that $f^{-1}(E)\in X$ for every $E\in A.$ show that $f^{-1}(F)\in X$ for any set F which belongs to the $\sigma-$algebra generated by A. (Hint: Use the preceding exercise.)

7.P. Let (X,X) be a measurable space and f be a real-valued function defined on X. show that f is X-measurable if and only if $f^{-1}(E)\in X$ for every Borel set E.

7.Q. Let (X,X) be a measurable space, f be an X-measurable function on X to R and let φ be a continuous function on R to R. Show that the composition $\varphi\circ f,$ defined by $(\varphi\circ f)(x)=\varphi[f(x)],$ is X-measurable.

(Hint: If φ is continuous, then $\varphi^{-1}(E)\in B$ for each $E\in B$.)

7.R. Let f be as in the preceding exercise and let Ψ be a Borel measurable function. Show that $\Psi\circ f$ is X-measurable.

7.S. Let f be a complex-valued function defined on a measurable space (X,X). Show that f is X-measurable if and only if

$$\{x\in X:a<\operatorname{Re}f(x)<b, c<\operatorname{Im}f(x)<d\}$$

Belong to X for all real numbers a, b, c, d. More generally, f is X-measurable if and only if $f^{-1}(G)\in X$ for every open set G in the complex plane C.

7.T. Show that sums, products, and limits of complex-valued measurable functions are measurable.

7.U. Show that a function f on X to R $\left(\text{or to}\bar{R}\right)$ is X-measurable if and only if the set A_α in Lemma 7.4(a) belongs to X for each rational number α; or, if and only if the set B_α in Lemma 7.4(b) belongs to X for each rational number α; etc.

7.V. A nonempty collection M of subsets of a set X is called a monotone class if, for each monotone increasing sequence (E_n) in M and each monotone decreasing sequence (F_n) in M, the sets

$$\bigcup_{n=1}^{\infty} E_n, \quad \bigcap_{n=1}^{\infty} F_n$$

Belong to M. Show that a $\sigma-$algebra is a monotone class. Also, if A is a nonempty collection of subsets of X, then there is a smallest monotone class class containing A. (This smallest monotone class is called the monotone class generated by A.)

7.W. If A is a nonempty collection of subsets of X, then the $\sigma-$algebra S generated by A contains the monotone class M generated by A. Show that the inclusion $A \subseteq M \subseteq S$ may be proper.

Check Your Progress

1. Prove: The following statements are equivalent for a function f on X to R:

(a) For every $\alpha \in R$, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$ belongs to X.

(b) For every $\alpha \in R$, the set $B_\alpha = \{x \in X : f(x) \le \alpha\}$ belongs to X.

(c) For every $\alpha \in R$, the set $C_\alpha = \{x \in X : f(x) \ge \alpha\}$ belongs to X.

(d) For every $\alpha \in R$, the set $D_\alpha = \{x \in X : f(x) < \alpha\}$ belongs to X.

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2. Prove: Let (f_n) be a sequence in $M(X, X)$ and define the functions

Notes

$$f(x) = \inf f_n(x), \quad F(x) = \sup f_n(x),$$

$$f^*(x) = \liminf f_n(x), \quad F^*(x) = \limsup f_n(x).$$

Then $f, F, f^*, \text{and } F^*$ belong to $M(X, X)$.

3. Prove: If f is a nonnegative function in $M(X, X)$, then there exists a sequence (φ_n) in $M(X, X)$ such that

(a) $0 \le \varphi_n(x) \le \varphi_{n+1}(x)$ for $x \in X, n \in N.$

(b) $0 \le \varphi_n(x) \le \varphi_{n+1}(x)$ for $x \in X, n \in N.$

(c) Each φ_n has only a finite number of real values.

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7.5 LET US SUM UP

1. A family X of subsets of a set X is said to be a $\sigma -$algebra (or a $\sigma -$ field) in case:

(i) ϕ, X belong to X.

(ii) If A belong to X, then the complement $\wp(A) = X / A$ belongs to X.

(iii) If (A_n) is a sequence of sets in X, then the union $\cup_{n=1}^{\infty} A_n$ belongs to X.

2. The following statements are equivalent for a function f on X to R:

(a) For every $\alpha \in R$, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$ belongs to X.

(b) For every $\alpha \in R$, the set $B_\alpha = \{x \in X : f(x) \le \alpha\}$ belongs to X.

(c) For every $\alpha \in R$, the set $C_\alpha = \{x \in X : f(x) \ge \alpha\}$ belongs to X.

(d) For every $\alpha \in R$, the set $D_\alpha = \{x \in X : f(x) < \alpha\}$ belongs to X.

3. Let (f_n) be a sequence in $M(X,X)$ and define the functions

$$f(x) = \inf f_n(x), \quad F(x) = \sup f_n(x),$$

$$f^*(x) = \lim\inf f_n(x), \quad F^*(x) = \lim\sup f_n(x).$$

Then $f, F, f^*, \text{and } F^*$ belong to $M(X,X)$.

4. If (f_n) is a sequence in $M(X,X)$ which converges to f on X, then f is in $M(X,X)$.

5. If f is a nonnegative function in $M(X,X)$, then there exists a sequence (φ_n) in $M(X,X)$ such that

(a) $0 \le \varphi_n(x) \le \varphi_{n+1}(x) \text{ for } x \in X, n \in N.$

(b) $0 \le \varphi_n(x) \le \varphi_{n+1}(x) \text{ for } x \in X, n \in N.$

(c) Each φ_n has only a finite number of real values.

7.6 KEY WORDS

Real valued function

Measurable function

Borel measurable function

7.7 QUESTIONS FOR REVIEW

1. Explain about theorems related to definitions related to measurable functions.
2. Explain about complex valued functions
3. Explain about functions between measurable spaces

7.8 SUGGESTIVE READINGS AND REFERENCES

1. A. Modern theory of Integration - Robert G.Bartle

2. The elements of Integration and Lebesgue Meassure

3. A course on integration- Nicolas Lerner

4. General theory of Integration- Dr. E.W. Hobson

5. General theory of Integration- P.Muldowney

6. General theory of functions and Integration- Angus E.Taylor

7.9 ANSWERS TO CHECK YOUR PROGRESS

1. See section 7.3
2. See lemma 7.9
3. See lemma 7.11